The Calculus of Variations and Variational Differential Geometry

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Preface

I wrote the following book as my senior thesis at the University of Washington which allows me to graduate with the distinction "graduated with departmental honors." Graduating with departmental honors is of course an achievement that anyone should be extremely proud of. The attachment of this distinction to my graduation record however is not the main reason for why I chose to write this book, rather it lies on a much more personal level which I think is important to explain.

During my second year at the University of Washington as an undergraduate I took an undergraduate course on topology and differential geometry with Professor Steve Mitchell. For the differential geometry course we used a textbook on differential geometry written by Manfredo P. Do Carmo, of which we covered the first four chapters. In approximately the same time period I also self-studied the calculus of variations using a book written on the subject by I.M. Gelfand and S.V. Fomin. Gelfand and Fomin's book is a wonderful book that I enjoyed very much and it gave me an incredible new power that allowed me to solve problems that I couldn't even dream of solving before.[1] The most important thing that it did however is it gave me a completely new way to approach differential geometry.

In that course on differential geometry we studied the differential geometry of two dimensional surfaces sitting in $\mathbb{R}^3$ and our approach did not use any kind of variational techniques. In my opinion however, approaching differential geometry without the use of variational techniques can produce a rather awkward point of view of the subject because it might obscure the true variational origin of many of the concepts studied in this field. For instance, after doing quite a bit of foundational work in that course we went ahead and defined geodesics on surfaces as unit speed curves $\gamma(t)$ that lie on the surface and that satisfy the property that their geodesics curvature is constantly equal to zero, which we wrote as:

$$k_g[\gamma] \equiv 0.$$

And then we went on to study many properties of geodesics. However, to me as a student back then this looked like a completely random definition of a type of curve that seemed to come out of nowhere. What was for example so special about these curves that warranted so much attention? The reason that I now imagine for why people first studied geodesics was not because they satisfied the above seemingly random definition, but because they had the property that they locally minimized arclength – a variational property. Indeed, the question of what is the shortest path on a surface is a natural question that people must have considered back then and the above

[1] I warn those who are thinking about reading Gelfand and Fomin's book though that the authors drop almost all rigor after page 27, which is 13% of the way towards the end of the book.

equation for the geodesics is in fact the Euler-Lagrange differential equation that seeks to find the extrema of the arclength functional. So the very defining equation for geodesics is of variational origin.

My thesis advisor argues otherwise on the point of geodesics. He argues that geodesics are a natural extension of the lines in the plane to curved surfaces because lines have the property that they do not curve in any direction. Indeed, suppose that you are a two-dimensional creature sitting on a two-dimensional curved surface with no notion of up or down (or more accurately: no notion of up or down in the normal direction to the surface). Then if you would analyze a curve that satisfies the property that its geodesic curvature is constantly equal to zero with your "curvature-o-meter," you would pick up zero curvature because the curve is not curving in any direction known to you.

During the summer between by second and third year at the University of Washington I was able to develop an approach to differential geometry using the calculus of variations that I learned from Gelfand and Fomin's book. And during that same time, I was also able to think of a new approach to the calculus of variations than what Gelfand and Fomin present in their book. Gelfand and Fomin take the approach of finding the extrema of a functional J over the space of functions by defining its variation $\varphi[h]$ as the linear functional that describes the functional J's linear behavior locally to a curve $y(x)$:

$$J[y+h]-J[y]=\varphi[h]+\varepsilon[h]\|h\|$$

where $\varepsilon[h]\to 0$ as $\|h\|\to 0$. The notion of locality in the domain of the functional is of course created by defining a norm on the space of functions. Then Gelfand and Fomin rigorously prove that the variation of the functional is zero at an extremum and then proceed to derive the Euler-Lagrange differential equation from there. This approach is very powerful; however, it does become a little bit more difficult to use when you get to more advanced topics such as variational problems with subsidiary conditions.

The approach that I thought of uses a concept that I like to call "flows." These flows that I construct deform the curve that you are trying find out whether it is an extremum of your functional in a way that is parametrized by one variable t. Then I take the composition of the functional with these flows and then use one variable calculus to arrive at necessary conditions for the extremums of functionals of certain forms. Not quite of course, there is a bit more work that goes into it. For example, in order to relate the extremums of the functions of t that you get from these sorts of compositions to the extremums of the functional that you are analyzing you have to show that these flows move through the space of curves continuously. And you have to consider a very large class of flows that pass through your curve. But the whole idea again is to convert the variational problem into something that is very intuitive: one variable calculus. And my approach has the benefit that it is easily applicable to more advanced variational problems such as variational problems with subsidiary conditions. Not only that, these flows allow one to study more global properties of functionals.

I do think that the reader should be aware that my thesis advisor was against my usage of the word "flow" to define such structures. He makes the valid point that flow is a concept that is already used in differential geometry to mean a completely different mathematical structure. His suggestion to me was to use words such as “variation” or “one parameter family” when talking about such structures. However, my argument is that I define flows in the field of the calculus of variations and not in differential geometry, and since the word “flow” doesn’t seem to have a standard meaning in the calculus of variations I feel like I am not causing much trouble by giving them the name “flows.” In addition, I really do like the imagery that the word “flow” brings to the concept since it accentuates the fundamental idea behind these structures in that they resemble a sort of movement in the space of functions. Personally when I think of flows, in my head I imagine the contour lines that resemble the movement of water in a river.

The calculus of variations also gave me the power to realize how the differential geometry that we did in that differential geometry course can be generalized into higher dimensional spaces. At one point during that summer I realized how to generalize the definitions of surface curvatures for $(n-1)$-dimensional surfaces sitting in $\mathbb{R}^n$ for general $n \in \mathbb{Z}_+$ and I knew that my generalizations were correct because they satisfied higher dimensional versions of many of the theorems that we covered in that differential geometry course.

A very large percentage of the mathematics that I do in this book represents what I discovered on my own during that summer using the two books that I mentioned above. I say this only for the purpose to warn you. The approach that is presented in this book passed through only the creative powers of one single mind and thus does not have the advantage of having been refined by centuries worth of mathematicians. But I do think that it might have the advantage of presenting a new or fresh perspective to the subject.

Since I like to discover things for myself, I don’t like to look in contemporary literature to learn about how modern mathematicians approach mathematical subjects before I give sufficient thought to them myself. Other than the two sources mentioned above, I haven’t looked at any other works in the fields of the calculus of variations and differential geometry and so for this reason I don’t know how others approach these two subjects. My thesis advisor has informed me however that my approach to this subject is completely standard and thus I don’t want anyone to get the idea that there are original results in this book. And I never suspected otherwise.

This book is structured in the following way. The first part, which is constituted of the first three chapters, develops the calculus of variations. The second part, which comprises the next three chapters, develops differential geometry using the calculus of variations developed in the first part. This way I think one can get a very systematic approach to these two fields and see how one is used to develop the other. I would also like to make the comment about what many people seem to describe as: my unorthodox writing style. My mathematical writing style sometimes tends to be a bit conversational in nature. Many people have advised me against writing in this manner, but the way I see it is that I love mathematics and with my writing style I strive to share my enthusiasm and passion for mathematics with all of my readers. For this reason, at times the language used in this book might more resemble a transcript of a lecture rather than a formal textbook. I just can’t do anything about that, it’s how I write. I have a sort of unconscious drive in me that strives to embed my character as a human being into all of my writing.

I will be submitting this book as my senior thesis at the end of this academic year. However, over the summer I would like to add additional material to this book that I wasn't able to include in this edition, including two new chapters on further interesting topics. In addition, I would probably like to rewrite some of the sections that explain surface parametrizations and surface curvatures to make them a bit more clear and concise. I will post this book and the most up to date edition on my personal website https://sites.math.washington.edu/~hgrebnev/D&Writings/Z_PDF_Documents_I/The Calculus of Variations and Variational Differential Geometry.pdf. If you do choose to read this book, please make sure that you have the latest edition of my book which you can find on the above website.

If you have any comments, would like to make suggestions about this book, or would like clarification on anything that I write in this book, please feel free to contact me at hgrebnev@uw.edu.

Haim R. Grebnev

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Please make sure that you are reading the most up to date version of this book, which you can find at https://sites.math.washington.edu/~hgrebnev/D&Writings/Z_PDF_Documents_I/The Calculus of Variations and Variational Differential Geometry.pdf.

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Notation:

- $[a, b]$ denotes a bounded closed interval. In other words when I write $[a, b]$ I mean the set of real numbers x such that $a \leq x \leq b$ and a and b are real numbers such that $a < b$.
- An integral of the form $\int_a^b$ means that a and b are finite real numbers such that $a < b$.
- For a mapping Φ, $\text{dom}(\Phi)$ means the domain of Φ.
- For a mapping Φ, $\text{ran}(\Phi)$ means the range of Φ.
- For a set $E \subseteq \mathbb{R}^n$, let ∂E denote its boundary.
- For a set $E \subseteq \mathbb{R}^n$, let E^{int} denote E's interior:

$$E^{int} = E \setminus \partial E$$

  In other words, E^{int} denotes everything that is in E but that isn't on the boundary of E. E^{int} is always an open set.
- For a function $h : E \subseteq \mathbb{R} \to \mathbb{R}$, $h^{(0)}(x)$ denotes the zeroth derivative, which is the function itself. In other words, $h^{(0)}(x) = h(x)$.
- For $x \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, let $B_r(x)$ denote the open ball of radius r centered at x:

$$B_r(x) = \{y \in \mathbb{R}^n : \|y - x\| < r\}.$$

- A subset $E \subseteq \mathbb{R}^n$ is called a "discrete set" if for every point in E there exists an open ball around it so that that point is the only point of E sitting in that ball. Symbolically this means that:

$$\forall x \in E \quad \exists r \in \mathbb{R}_+, \qquad B_r(x) \cap E = \{x\}.$$

  People often call the points of a discrete set "isolated points" since qualitatively one can describe them as being isolated by a positive distance from the rest of the set.
- When a parametrized curve $\gamma : [t_0, t_1] \to \mathbb{R}^n$ is called **non-singular**, this means that $\gamma'(t)$ never vanishes (meaning $\gamma'(t)$ is never the zero vector).
- When a real-valued function $F : \mathbb{R}^n \to \mathbb{R}$ is called **non-singular**, this means that its gradient ∇F never vanishes (meaning ∇F is never the zero vector).
- A **plane** in $\mathbb{R}^n$ will mean a linear subspace of dimension $n - 1$. Some people use the word hyperplane for this.
- If $U \subseteq \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, then $C^k[U, \mathbb{R}^m]$ is the set of k-times continuously differentiable functions of the form $\Psi : U \to \mathbb{R}^m$.

Chapter 1: Basic Elements of the Calculus of Variations

"It is just a matter of unraveling the definitions." – Steve Mitchell

Section 1: Introduction

With this chapter we will begin the study of the calculus of variations. We might as well start off with the question: what in the world is the calculus of variations? The calculus of variations is an important branch of calculus that studies the question of how to find extrema of quantities that depend on functions. Usual differential calculus studies the question of how to find extrema of quantities that depend on several variables (i.e. functions of the form $F : \mathbb{R}^n \to \mathbb{R}$), while the calculus of variations studies the question of how to find the extrema of functions of the form $J : S \to \mathbb{R}$ where S is some set of functions. The class of such functions $J : S \to \mathbb{R}$ where S is some set of functions has been given a special name: they're called "**functionals**." So in essence, we can say that the calculus of variations is a branch of calculus that studies the question of how to find the extrema of functionals.

Functionals play many important roles in mathematics and physics since many useful quantities can be represented as mappings from a space of functions to the real line. Let us look at some examples of functionals:

Example 1.1.1: Let the functional $J : S \to \mathbb{R}$ be given by:

$$J[y(x)] = y(0)$$

where the domain S of this functional is the set of all real-valued continuous functions $y(x)$. Here J takes any real-valued continuous function $y(x)$ and returns its value at $x = 0$.

Example 1.1.2: Let the functional $J : S \to \mathbb{R}$ be given by:

$$J[y(x)] = \max_{x \in [a,b]} \{y(x)\}$$

where the domain S of this functional is the set of all continuous functions $y(x)$ over the compact interval $[a, b]$. For any continuous function $y(x)$ this functional returns the maximum value that $y(x)$ attains on the interval $[a, b]$ (that such a maximum number exists follows from the Extreme Value Theorem). This functional is often used in mathematics (as we will soon) in order to define a norm on the space of continuous functions over a compact interval.

Example 1.1.3: Let the functional $J : S \to \mathbb{R}$ be given by

$$J[y(x)] = \int_a^b \sqrt{1 + \big(y'(x)\big)^2} dx$$

where the domain S of this functional is the set of all continuously differentiable functions $y(x)$ defined on the compact interval $[a, b]$ (we have to restrict the domain of J to all continuously differentiable functions $y(x)$ or else the above integral might not make sense). The above integral is the arc-length integral. So basically, the above functional takes a function $y(x)$ and outputs its arc-length over the compact interval $[a, b]$. This is called the arc-length functional.

Example 1.1.4: Let S be the set of all decreasing real-valued differentiable functions $y(x)$ over the compact interval $[a, b]$ that satisfy the boundary conditions:

$$y(a) = A \quad and \quad y(b) = B$$

for some fixed numbers $A > B$. For any function $y(x)$ in this set let the functional $J : S \to \mathbb{R}$ return the time for a point particle to slide down the curve $y(x)$ under the influence of gravity (disregarding friction) from the point (a, A) to the point (b, B) (here "curve" means "function of one variable" such as $y(x)$).

Note 1.1.5: As we did in the previous example, we will often interchange the words "curve" and function of one variable. If we were to be more precise we should be saying the "curve generated by the graph of the function of $y(x)$ (our function of one variable)." But as a matter of convenience we will be loose with this terminology. If we want to make a distinction between a general curve and a graph of a function, we will sometimes say "parametrized curve" instead of just plain old "curve," but this doesn't matter too much at the current moment since we won't encounter parametrized curves for some time.

Example 1.1.6: As a more general example, we can write down a functional in the form:

$$J[y(x)] = \int_a^b F\big(x, y(x), y'(x)\big) dx$$

where $F(u, v, w) : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function and the domain of the above functional is the set of all continuously differentiable functions $y(x)$ over the compact interval $[a, b]$. For example, the functional in Example 1.1.4 is the special case $F(u, v, w) = \sqrt{1 + w^2}$.

With these examples in mind, what are some problems that the calculus of variations studies how to solve? Here are some examples:

Example 1.1.7: (The planar arclength minimizing curve problem) Of all the differentiable curves $y(x)$ that connect two fixed points (a, A) and (b, B), where $a < b$, which has the smallest arc-length? To reformulate this problem in terms of functionals, let S be the set of all differentiable curves $y(x)$ over the compact interval $[a, b]$ that satisfy the boundary conditions:

$$y(a) = A \quad and \quad y(b) = B$$

(remember, as mentioned in Note 1.1.5 here "curve" means "function of one variable"). Then this problem is the same thing as finding the minimum of the arc-length functional $J : S \to \mathbb{R}$ given by:

$$J[y] = \int_a^b \sqrt{1 + y'^2} dx$$

(I was too lazy – as I will often be – to write the argument of y in the above equation). So in other words we have to find the curve $y(x)$ in S that minimizes the above integral. Of course, the reader should immediately guess that the answer to the above problem is the straight line between the points (a, A) and (b, B).

Example 1.1.8: (The Brachistochrone Problem) Of all the decreasing differentiable curves $y(x)$ that connect the points (a, A) and (b, B) (where $b > a$ and $A > B$), which curve has the smallest transit time for a point particle to slide down the curve from the point (a, A) to the point (b, B) under the influence of gravity (disregarding friction)? Again let us reformulate the problem in terms of functionals. Let S be the set of all decreasing differentiable curves $y(x)$ over the compact interval $[a, b]$ that satisfy the boundary conditions:

$$y(a) = A \quad and \quad y(b) = B.$$

As we will show later, the transit time for a point particle to slide down the curve from the point (a, A) to the point (b, B) under the influence of gravity (disregarding friction) is given by:

$$J[y] = \int_a^b \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}} dx$$

where g is the acceleration of gravity constant towards earth ($g = 9.8\ meters/second^2$). With this we get that the Brachistochrone problem can be reformulated as the problem of finding the curve that is the minimum of the above functional. The curve $y(x)$ in S that minimizes the above integral is called the Brachistochrone curve. We will later find an explicit equation for it.

Example 1.1.9: A more general type of problem that the calculus of variations studies how to solve is the minimization of a functional $J : S \to \mathbb{R}$ of the form:

$$J[y] = \int_a^b F(x, y, y') dx$$

over some domain S. The previous two examples were of this form where in the minimizing curve problem we had that $F(u,v,w) = \sqrt{1+w^2}$ and in the Brachistochrone problem $F(u,v,w) = \sqrt{1+w^2}/\sqrt{2gv}$.

Unfortunately, in the calculus of variations there are no standard ways developed for finding the extrema of all types of functionals. In fact, the relative scope of functional extremum problems that the calculus of variations can currently solve is rather small. The functional extremum problems that the calculus of variations primarily focuses on solving are the problems of finding the extrema of functionals that can be represented in the form (or of similar form):

$$J[y] = \int_a^b F(x,y,y')dx.$$

We've seen at least two examples of such functionals so far. Fortunately, many of the functional extremum problems that arise in mathematics and physics fall into this category and so without further delay we begin the study of how to find the extrema of functionals in this class.

Section 2: Distances and Flows between Curves

One of the questions that might arise is, why is the calculus of variations called the "calculus of variations." Looking at the type of problems that were described in the previous section that the calculus of variations studies, this field does indeed look like calculus. I mean just like calculus, the calculus of variations studies the problem of finding extremums of certain types of functions (functionals to be precise). But why is it called the calculus of "variations?" What exactly is "varying" in our case? Well the short answer is that just like we did in calculus, the calculus of variations approaches the problem of finding extremums of functionals from a local point of view. In other words, in order to find extremums of functionals the calculus of variations first looks for "local extremums" of functionals. And what does it mean for a curve, say $Y(x)$, to be an extremum of a functional $J[y]$. It means that locally to $Y(x)$ the values of J at curves near $Y(x)$ is either always bigger or always less than the values of J at $Y(x)$. Thus the approach that the calculus of variations must take in order to find the local extremums of functionals requires one to be able to analyze the value of the functional J at nearby curves to the extremum. This analysis is classically done by "varying" the potential extremum curve just a little bit and looking at how the values of the functional J change as the you are doing the variation. Hence the name, "the calculus of variations."

When I first heard this idea of varying curves just a little bit and analyzing the values of the functional at those new curves as you do the variation, I was completely overwhelmed. I wondered how on earth are you going to try varying in all possible ways? And I was right to be overwhelmed since the space of functions of one variable/curves is huge! But there are ways to overcome this obstacle and one way is to define derivatives of functionals over the space of curves. Another way is through an idea called "flows." Both approaches work equally well, and we will take the approach that uses the second concept, the concept of "curve flows." Let's begin.

Calculus of variations borrows its main ideas directly from ordinary differential calculus. How does one for example in normal differential calculus find the extrema of a function of the form $F : E \subseteq \mathbb{R}^n \to \mathbb{R}$? The main idea is to first find all points $x \in E$ in the domain of F that are either local minima or local maxima of the function. In more symbolic terms, the first step is to find all points $x \in E$ such that either of the following happens:

$$\exists \delta > 0 \quad \forall h \in E : \|h - x\| \leq \delta, \;\; F(h) \geq F(x)$$

or

$$\exists \delta > 0 \quad \forall h \in E : \|h - x\| \leq \delta, \;\; F(h) \leq F(x).$$

If the first case happens, then x is a local minimum. If the second case happens, then x is a local maximum. After finding all such points, we try to see if any of these points are global extrema of the function F. We only look for global extrema among these sets of points (local extrema that is) because all global extrema are necessarily local extrema. Sometimes functions may not have global extrema. But if they do, they must belong to the set of local extrema.

The idea in the calculus of variations is exactly the same. In order to find the global extrema of a functional J, we first find all of its local extrema. Easier said than done since in order to even define "local extrema" of a functional we must first define some sense of "closeness" between two elements of a functional's domain. For this purpose, we will need to define a distance function in the space of functions.

Before we proceed any further, let us introduce a notation.

Notation 1.2.1: Suppose that $E \subseteq \mathbb{R}^n$. Then for any positive integer k, let $C^k[E]$ denote the set of all functions of the form $F : E \to \mathbb{R}$ such that all their kth order and lower partials are continuous on E. A more symbolic way to say this is that a function $F(x_1, x_2, \dots, x_n) \in C^k[E]$ if for any set of non-negative integers $\{a_1, a_2, \dots, a_n\}$ such that $a_1 + a_2 + \cdots + a_n \leq k$, the partial:

$$\frac{\partial^{a_1 + a_2 + \cdots + a_n} F}{\partial x_1^{a_1} \partial x_2^{a_2} \dots \partial x_n^{a_n}} = \frac{\partial^{\sum_{j=1}^n a_j} F}{\prod_{j=1}^n \partial x_j^{a_j}}$$

is continuous on E. Let $C^0[E]$ denote the set of all continuous functions on E and let $C^\infty[E]$ denote the set of all functions such that all of their partials are continuous.

There is a technicality that we need to address. For a function $F \in C^k[E]$, when we say that $\frac{\partial^{\sum_{j=1}^n a_j} F}{\prod_{j=1}^n \partial x_j^{a_j}}$ is continuous at a boundary point $x \in \partial E$ of E we mean that F can be extended to a function $\tilde{F} \in C^k[R]$ where $E \subset R$ and $x \in R^{int}$ such that $\frac{\partial^{\sum_{j=1}^n a_j} \tilde{F}}{\prod_{j=1}^n \partial x_j^{a_j}}$ is continuous at x. In other words, we mean that we can extend the function F onto a bigger set that contains the point x in its interior and the same partial of the extended function is continuous at that point. This is a small technicality that's needed since technically partials of functions are defined only on open sets.

Note 1.2.2: It should also be noted that in order to check that a function F belongs to $C^k[R]$, it is sufficient to check that all of F's kth order partials are continuous. From multivariable calculus we know that this will imply that all of F's lower order partials are also continuous.

The above notation will simply make it easier for us to write down the domain of many of the functionals that we will be encountering.

Now where were we. Ah yes, we said that we needed to define a notion of distance in a space of functions. For this we make our first definition.

Definition 1.2.3: *Let E be a compact subset of $\mathbb{R}$. The way we will define the* ***norm*** *of a function $y(x) \in C^n[E]$ is by:*

$$\|y\| = \sum_{k=0}^{n} \max_{x \in E}\left\{\left|y^{(k)}(x)\right|\right\}$$

(We look at compact subsets E of $\mathbb{R}$ so as to guarantee by the Extreme Value Theorem that all of the above $\max_{x \in E}\left\{y^{(k)}(x)\right\}$ actually exist).

It is easy to check that the above definition agrees with the usual notion of a "norm" on a space (meaning that it satisfies the 4 axioms of a norm). There are of course other ways to define a norm of a function $y(x) \in C^n[E]$, but as we will see in the rest of this book this turns out to be a very convenient norm to work with. With this definition of a norm of a function we can now define the distance between two functions $y_1(x)$ and $y_2(x)$.

Definition 1.2.4: *Let E be a compact subset of $\mathbb{R}$. For any two curves $y_1(x), y_2(x) \in C^n[E]$, we define the* ***distance*** *between them by:*

$$\|y_1 - y_2\| = \sum_{k=0}^{n} \max_{x \in E}\left\{\left|y_1^{(k)}(x) - y_2^{(k)}(x)\right|\right\}.$$

Again one can check that the above definition of "distance" does agree with the usual notion of a "distance/metric" on a space (meaning that it satisfies the 4 axioms of a metric/distance function).

At first glance one might say that the above definition of distance between two curves is pretty awkward. Why do we for example look at all of the possible derivatives of these two curves in order to determine how close they are? I mean, can we just look at how far they are in value and define that as their distance. The following example will help answer this question by showing that the more derivatives we look at, the more accurate sense we get at how close two curves resemble each other.

Example 1.2.5: Let us take the two curves $y_1, y_2 : [0, 1] \to \mathbb{R}$ defined by:

$$y_1(x) = x$$

$$y_2(x) = \frac{1}{100}\sin(100x) + x$$

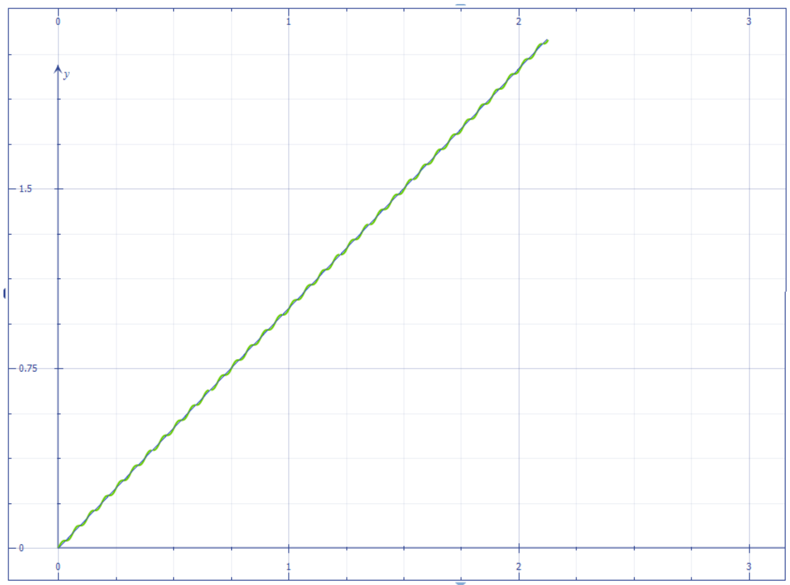


Figure 1: Graph of y_1 (dark blue) and y_2 (green)

Notice that the distance between these two curves when viewed as curves sitting in the space $C^0[0,1]$ is:

$$\|y_1 - y_2\| = \max_{x\in[a,b]}\{|y_1(x) - y_2(x)|\} = \frac{1}{100} = 10^{-2},$$

a pretty small number. While the distance between them as viewed as curves sitting in the space $C^1[0,1]$ is:

$$\|y_1 - y_2\| = \max_{x\in[a,b]}\{|y_1(x) - y_2(x)|\} + \max_{x\in[a,b]}\{|y_1'(x) - y_2'(x)|\} = \frac{1}{100} + 1 \approx 1,$$

a fairly big number. So from the perspective of continuous functions ($C^0[0,1]$) these curves are really close because the distance between them is a mere 10^{-2}. However, the distance between them from the perspective of continuously differentiable functions ($C^1[0,1]$) is pretty big, nearly a whole big 1. This make sense since in value over the interval $[0,1]$ the functions do not stray too far from each other. On the other hand, in terms of their tangent forms (first derivatives) these two curves don't resemble each other at all. One of them is really straight while the other one is very wiggly.

This example illustrates the point that the more derivatives you look at the more accurate description you get of how close two curves resemble each other. This is why the definition of the distance between two curves, being a measure of how much they resemble each other, looks at the maximum number of possible differences of derivatives between two curves in order to tell how far apart they are.

Now that we have a notion of distance between curves, we are now in a position to define what it means for a curve to be a local minimum or local maximum of a functional. It's defined in an exactly analogous way as it is done for real-valued functions of several variables.

Definition 1.2.6: *Suppose that we have a functional $J : S \subseteq C^k[E] \to \mathbb{R}$ where E is some compact subset of $\mathbb{R}$. Then $Y(x) \in S$ is called a* ***local minimum*** *of the functional J if:*

$$\exists \delta > 0 \quad \forall h \in S : \|h - Y\| \leq \delta, \;\; J[h] \geq J[Y]$$

and $Y(x) \in S$ is called a ***local maximum*** *of J if:*

$$\exists \delta > 0 \quad \forall h \in S : \|h - Y\| \leq \delta, \;\; J[h] \leq J[Y].$$

The statements in the above definition are actually pretty intuitive. For example, in the definition of $Y(x)$ being a local minimum the above logical statement is merely saying that there exists a distance $\delta > 0$ such that the value of the functional at any curve $h \in S$ in the domain that is within a distance of δ from Y is going to be bigger that $J[Y]$. This is indeed what a local minimum should be and that is why we have defined it so in the above definition. The same goes for the local maximum definition.

Definition 1.2.7: *Suppose that we have a functional $J : S \subseteq C^k[E] \to \mathbb{R}$ where E is some compact subset of $\mathbb{R}$. Then $Y(x) \in S$ is called a* ***global minimum*** *of the functional J if:*

$$\forall h \in S, \qquad J[h] \geq J[Y]$$

And $Y(x) \in S$ is called a ***global maximum*** *of the functional J if:*

$$\forall h \in S, \qquad J[h] \leq J[Y]$$

In our quest to gain a better understanding and handle on the space of curves, we defined a sense of distance on the space of curves $C^n[E]$. With this we were able to define local extrema of functionals. However there is one more key ingredient that we need in order to get a sufficient handle on the space of curves so that we can finally start actually looking for the local extrema of functionals. The key ingredient is to define a sense of movement in the space of curves.

In one variable differential calculus, we were able to find the derivative of a differentiable function $f(x)$ at the point x_0 by letting x approach the point x_0 on the x-axis and taking the limit:

$$f'(x) = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right).$$

We were also able to do a similar thing in multivariable calculus with gradients. In the calculus of variations we will want to do a similar thing, but in order to do that we need to define some sense of movement in the space of curves (just like how we had x approach x_0 in the above calculation/definition of $f'(x)$). For this we define an n-smooth linear flow:

Definition 1.2.8: *An* ***n-smooth linear flow*** *is a function $\Lambda : [a, b] \times [t_0, t_1] \to \mathbb{R}$ of the form:*

$$\Lambda(x, t) = y_1(x) + y_2(x) \cdot t$$

where $y_1(x)$ and $y_2(x)$ are two functions (or "curves", since we often use the word "curve" for "real-valued function of one variable") such that $y_1, y_2 \in C^n[a, b]$.

We will not think of n-smooth linear flows as merely functions of two variables with the above form. For every fixed time $t_2 \in [t_0, t_1]$, we look at $\Lambda(x, t_2)$ as a curve in $C^n[a, b]$. Thus as the time variable t varies across the interval $[t_0, t_1]$, the curve $\Lambda(x, t)$ changes/"flows" as t changes. In other words, as t changes the "curve" $\Lambda(x, t)$ moves through the space $C^n[a, b]$. Thus n-smooth linear flows give us the notion that we wanted of a type of movement in the space $C^n[a, b]$. We will look at an example of n-smooth linear flows in Example 1.2.9.

The "linear" part in the name "n-smooth linear flows" comes from the fact that the above equation for Λ resembles an equation for a line in $\mathbb{R}^n$:

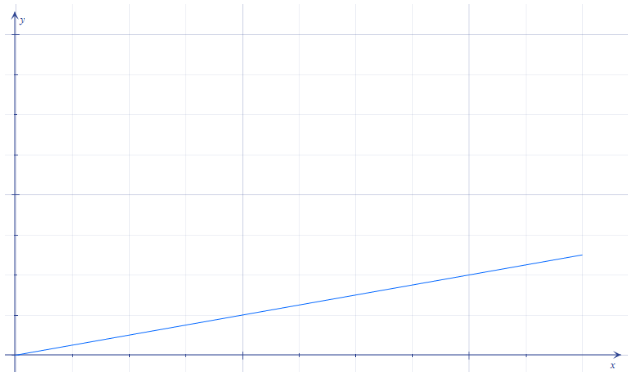
$$a + bt$$

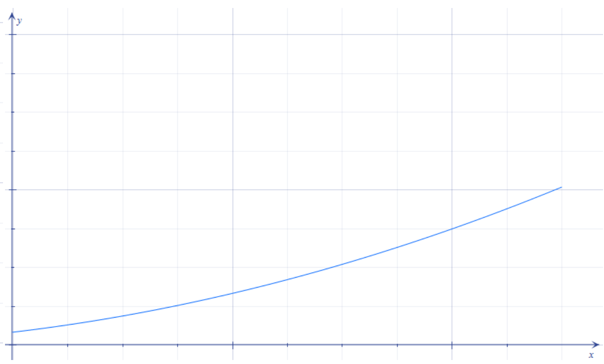
where $a, b \in \mathbb{R}^n$. Thus, we can think of n-smooth linear flows as lines in the space $C^n[a, b]$.[2]

Example 1.2.9: Take the 3-smooth linear flow $\Lambda : [0, 5] \times [0, 1] \to \mathbb{R}$ defined by:

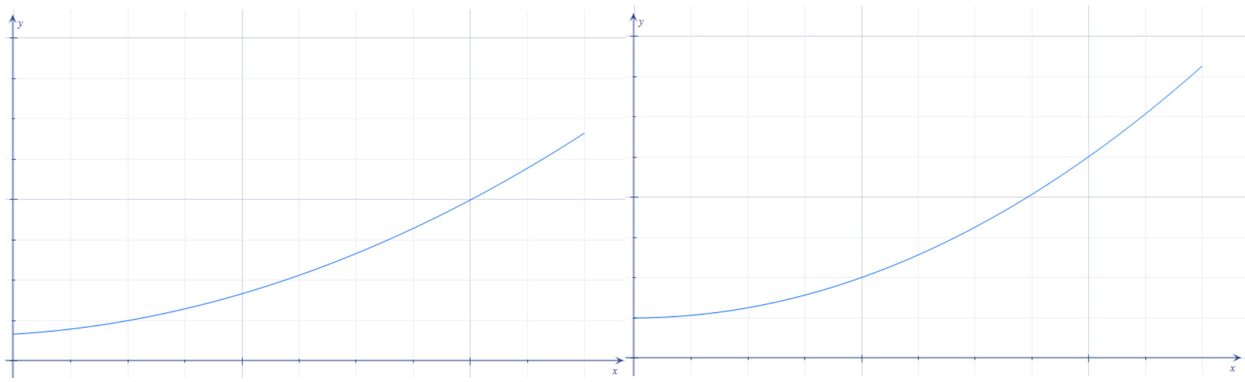
$$\Lambda(x, t) = 2x + t(4 + x^2 - 2x)$$

At time $t = 0$ we see that the curve $\Lambda(x, 0)$ is the line $2x$ over the interval $[0, 5]$. At time $t = 1$ the curve $\Lambda(x, 1)$ is the curve $4 + x^2$ over $[0, 5]$. Indeed, we can visualize the flow Λ by plotting $\Lambda(x, t)$ for several values of t. The following graphs are plots of the curves $\Lambda(x, 0)$, $\Lambda(x, 0.33)$, $\Lambda(x, 0.66)$, and $\Lambda(x, 1)$ respectively.





[2] In fact, for the curious reader this should suggest that it's possible to define more general types of flows, not just "lines" in the space $C^n[a, b]$.



As one can see, over time $\Lambda(x,t)$ goes from looking like the straight curve $2x$ to looking like the curve $4+x^2$. In this example, we could informally say that the flow $\Lambda(x,t)$ flows from the curve $2x$ to $4+x^2$ in the space $C^3[0,5]$.

While write down n-smooth linear flows $\Lambda(x,t)$ as functions of two variables that map from the set $[a,b]\times[t_0,t_1]$ to $\mathbb{R}$, we rather interpret them as functions of t that map from the time interval $[t_0,t_1]$ to the space of curves. In other words, for every fixed $t\in[t_0,t_1]$ we think of $\Lambda(x,t)$ as a curve of the variable x. Seeing this interpretation of $\Lambda(x,t)$ you might then ask: why don't we instead just write linear flows Λ as maps from the time interval $[t_0,t_1]$ to the space of curves $C^n[a,b]$. After all, if we think of linear flows Λ as just a bunch of curves that depend on the time variable $t\in[t_0,t_1]$, why don't we just write n-smooth linear flows as maps of the form $\Lambda:[t_0,t_1]\to C^n[a,b]$. The answer is that we do, and we will do this using the canonical form of the linear flow.

Definition 1.2.10: *Let $\Lambda:[a,b]\times[t_0,t_1]\to\mathbb{R}$ be an n-smooth linear flow as defined in Definition 1.2.8. Then the* ***canonical form*** *of $\Lambda(x,t)$ is the function:*

$$\widetilde{\Lambda}:[t_0,t_1]\to C^n[a,b]$$

defined by:

$$\widetilde{\Lambda}(t)=\Lambda(x,t)$$

where in this case $\Lambda(x,t)$ on the right-hand side denotes the curve of the variable x for every fixed t.

Example 1.2.11: Take the 3-smooth linear flow in example 1.2.9. What is its canonical form $\widetilde{\Lambda}(t)$? Figuring out the canonical form is merely a question of notation/function form. For each $t\in[0,1]$, we have that $\widetilde{\Lambda}(t)$ is the curve:

$$\Lambda(x,t)=2x+t(4+x^2-2x)$$

where t is held constant. So we have that:

$$\widetilde{\Lambda}(t)=2x+t(4+x^2-2x)\in C^n[a,b].$$

The difference between an n-smooth linear flow $\Lambda(x,t)$ as defined in Definition 1.2.8 and its canonical form $\widetilde{\Lambda}$ is really just a matter of reformulating the same thing in different function form/notation. Both forms are useful. The classical two variable notation allows us to do more calculus on the flow such as considering partials of the form $\frac{\partial^{k+j}\Lambda}{\partial x^k \partial t^j}$, which we will need in our derivation of the Euler-Lagrange Differential Equation (Theorem 1.3.3). The canonical form $\widetilde{\Lambda}$ however lends itself more to the study of how the linear flow moves through the space $C^n[a,b]$. In other words, the canonical form allows for a more topological study of linear flows rather than a calculus one.[3]

One very important topological property of n-smooth linear flows is that they flow/move through the space $C^n[a,b]$ continuously. This is stated precisely in the following theorem:

Theorem 1.2.12: *Let* $\Lambda : [a,b] \times [t_0, t_1] \to \mathbb{R}$ *be an n-smooth linear flow and let* $\widetilde{\Lambda} : [t_0, t_1] \to C^n[a,b]$ *be its canonical form. Then* $\widetilde{\Lambda}(t)$ *is a continuous function.*

Proof: The proof of this theorem isn't difficult and it's merely a task of unwinding the definition of the norm on the space $C^n[a,b]$ and what it means to be a continuous function. Intuitively the reason for why this theorem is true is that at every point in time $t \in [t_0, t_1]$, we can find an upper bound for how fast the first n derivatives of $\widetilde{\Lambda}(t)$ change with time. Since the norm of a curve in $C^n[a,b]$ is defined using the first n derivatives of the curve, we can get an upper bound for how far a linear flow moves in the space $C^n[a,b]$ in a small amount of time. Now let's do this rigorously. To prove this theorem with rigor is just a game of inequalities. What do we have to do in order to prove that $\widetilde{\Lambda}$ is a continuous function? We have to prove that:

$$\forall t_2 \in [t_0, t_1] \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall t \in [t_0, t_1] : |t - t_2| \leq \delta, \qquad \left\|\widetilde{\Lambda}(t) - \widetilde{\Lambda}(t_2)\right\| \leq \varepsilon.$$

This merely comes from the ε-δ definition of the continuity of a function at every point of its domain. Pick any $t_2 \in [t_0, t_1]$ and any $\varepsilon > 0$. We want to prove the existence of such a $\delta > 0$. For any $t \in [t_0, t_1]$, we have that:

$$\left\|\widetilde{\Lambda}(t) - \widetilde{\Lambda}(t_2)\right\| = \sum_{k=0}^{n} \max_{x \in [a,b]} \left\{ \left| \frac{d^k}{dx^k}\left(\widetilde{\Lambda}(t)\right) - \frac{d^k}{dx^k}\left(\widetilde{\Lambda}(t_2)\right) \right| \right\}.$$

Since $\widetilde{\Lambda}(t) = \Lambda(x,t)$ for all $x \in [a,b]$ and $t \in [t_0, t_1]$ (see Definition 1.2.10), we get that for $\forall k \in \{0,1,\dots,n\}$,

$$\frac{d^k}{dx^k}\left(\widetilde{\Lambda}(t)\right) = \frac{\partial^k \Lambda(x,t)}{\partial x^k}.$$

Thus, we can rewrite the equation before in a more convenient form:

$$\left\|\widetilde{\Lambda}(t) - \widetilde{\Lambda}(t_2)\right\| = \sum_{k=0}^{n} \max_{x \in [a,b]} \left\{ \left| \frac{\partial^k \Lambda(x,t)}{\partial x^k} - \frac{\partial^k \Lambda(x,t_2)}{\partial x^k} \right| \right\}.$$

Since Λ is a linear flow, it can be written down in the form:

[3] Topology is a branch of mathematics that studies continuous maps. Knowledge of topology isn't required here.

$$\Lambda(x,t) = y_1(x) + y_2(x) \cdot t$$

for some $y_1, y_2 \in C^n[a,b]$. Plugging this into the previous equation gives us that:

$$\left\|\widetilde{\Lambda}(t) - \widetilde{\Lambda}(t_2)\right\| = \sum_{k=0}^{n} \max_{x\in[a,b]} \left\{ \left| (t - t_2) \frac{d^k y_2}{dx^k}(x) \right| \right\} = |t - t_2| \sum_{k=0}^{n} \max_{x\in[a,b]} \left\{ \left| \frac{d^k y_2}{dx^k}(x) \right| \right\}.$$

If suddenly discover that $y_2 \equiv 0$, then notice that this equation trivially implies that $\widetilde{\Lambda}$ is continuous (because then $\sum_{k=0}^{n} \max_{x\in[a,b]} \left\{ \left| \frac{d^k y_2}{dx^k}(x) \right| \right\} = 0$ and so $\left\|\widetilde{\Lambda}(t) - \widetilde{\Lambda}(t_2)\right\|$ would always be zero and thus always less than ε). So, suppose that $y_2 \not\equiv 0$. Then $\sum_{k=0}^{n} \max_{x\in[a,b]} \left\{ \left| \frac{d^k y_2}{dx^k}(x) \right| \right\} \neq 0$ and so notice that if we now set $\delta = \varepsilon / \sum_{k=0}^{n} \max_{x\in[a,b]} \left\{ \left| \frac{d^k y_2}{dx^k}(x) \right| \right\}$, we then have what we want: for $\forall t \in [t_0, t_1] : |t - t_2| \leq \delta$,

$$\left\|\widetilde{\Lambda}(t) - \widetilde{\Lambda}(t_2)\right\| = |t - t_2| \sum_{k=0}^{n} \max_{x\in[a,b]} \left\{ \left| \frac{d^k y_2}{dx^k}(x) \right| \right\} \leq \delta \sum_{k=0}^{n} \max_{x\in[a,b]} \left\{ \left| \frac{d^k y_2}{dx^k}(x) \right| \right\}$$

$$= \frac{\varepsilon}{\sum_{k=0}^{n} \max_{x\in[a,b]} \left\{ \left| \frac{d^k y_2}{dx^k}(x) \right| \right\}} \sum_{k=0}^{n} \max_{x\in[a,b]} \left\{ \left| \frac{d^k y_2}{dx^k}(x) \right| \right\} = \varepsilon.$$

And so this $\delta > 0$ works. With this we have proved the theorem.

■

Section 3: First Results

With our newly acquired understanding and handle on the space of curves that we got in the previous section, we can now arrive at some of our first results. The following is a fairly straightforward but very powerful lemma.

The proof of the following lemma comes from Gelfand and Fomin's book on the calculus of variations.

Lemma 1.3.1: *Let n be a fixed non-negative integer. Suppose that $\alpha(x) \in C^0[a,b]$ is a continuous function such that for any $h \in C^n[a,b]$ that satisfies the boundary conditions:*

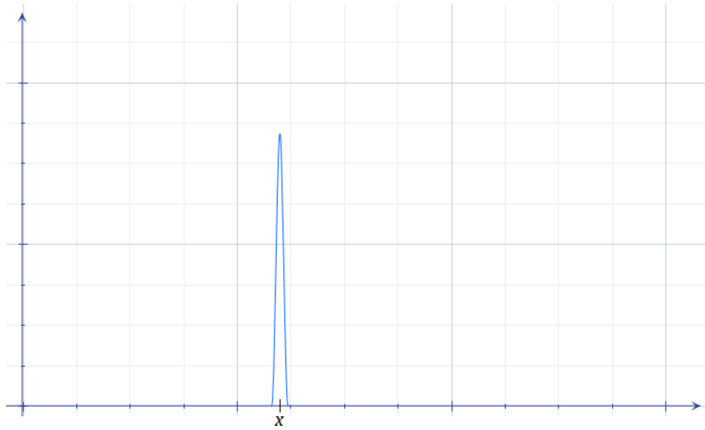
$$h(a) = h(b) = 0,$$

the following integral is equal to zero:

$$\int_a^b \alpha(x) h(x) dx = 0.$$

Then $\alpha(x) = 0$ on all of $x \in [a,b]$.

Proof: The whole idea behind this lemma is that if α was nonzero at some point x, then by α's continuity we would know that locally to x, α would also be nonzero. From there we would be able to construct a function $h \in C^n[a,b]$ that is zero everywhere except near x where h spikes up:



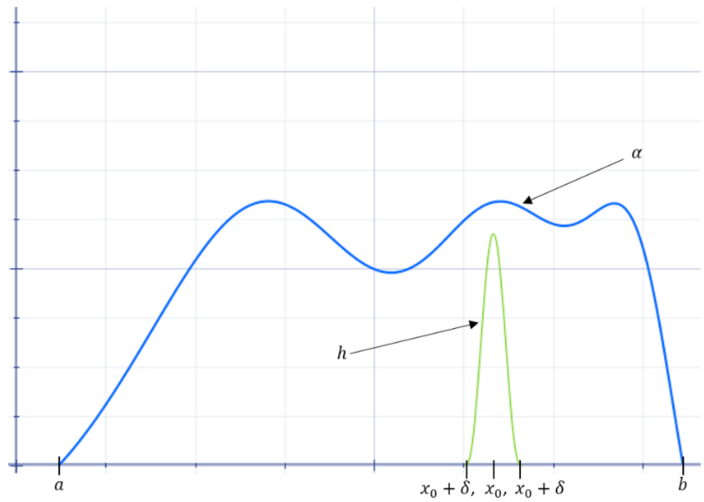
Then intuitively in terms of area we could see that with this h the integral $\int_a^b \alpha(x)h(x)dx$ would be nonzero, which would be a contradiction since we said that for all h that satisfy the above boundary conditions the integral $\int_a^b \alpha(x)h(x)dx = 0$. Now let us do this formally and rigorously. If you, dear reader, can or want to think of a way to rigorize and formalize the above arguments into a full proof of the lemma yourself, then I urge you to do so and skip the rest of this proof (it is much better to think of your own proofs to theorems rather than reading them). If not, here we go!

We will prove this lemma by contradiction. Suppose not. Suppose that it was not true that $\alpha(x) = 0$ on all of $[a,b]$. Then $\alpha(x)$ is non-zero somewhere in $[a,b]$. In fact, I claim that this means that we can always find $x_0 \in (a,b)$ (the interior of the interval of $[a,b]$) such that $\alpha(x_0) \neq 0$. Why is this so? Well, by the fact that it is not true that $\alpha(x) = 0$ on all of $[a,b]$, we get that there exists a point $x_1 \in [a,b]$ such that $\alpha(x_1) \neq 0$. If x_1 is in the interior of the interval, then x_1 is the x_0 that we want. Now if it happened that x_1 landed on the boundary of the interval $[a,b]$, then by the continuity of α we can always choose x_0 that is in the interior of the interval $[a,b]$ and that is close enough to x_1 such that $\alpha(x_0) \neq 0$. So we have proved that there exists a point $x_0 \in (a,b)$ such that $\alpha(x_0) \neq 0$.

Let us suppose that $\alpha(x_0) > 0$ (the case $\alpha(x_0) < 0$ is treated in almost exactly the same way) Pick any $\varepsilon > 0$ such that $\varepsilon < \alpha(x_0)$ (my professors' favorite such ε was always $\frac{\alpha(x_0)}{2}$ for some reason). By the continuity of α at x_0 we know that there exists a δ such that the values of $\alpha(x)$ in the interval $[x_0 - \delta, x_0 + \delta]$ will be bigger than or equal to ε and such that $[x_0 - \delta, x_0 + \delta]$ will still lie in the interval $[a,b]$. Now define $h(x)$ by:

$$h(x) = \begin{cases} 0 \quad if \quad x \notin [x_0 - \delta, x_0 + \delta] \\ \frac{1}{\delta^2}\left[\big(x - (x_0 - \delta)\big)\big((x_0 + \delta) - x\big)\right]^{n+1} \quad if \quad x \notin [x_0 - \delta, x_0 + \delta] \end{cases}$$

This is a curve that is zero everywhere except it spikes up on the interval $[x_0 - \delta, x_0 + \delta]$. If one wants to get a feel for how the spike in the above h looks like, I recommend graphing it for various values of x_0, δ, and n. It's a useful technique for writing down an equation for the graph of a spike. It's not hard to see that $h \in C^n[a,b]$ and since $[x_0 - \delta, x_0 + \delta] \subseteq [a,b]$ (by construction) we get that our $h(x)$ does indeed satisfy the boundary conditions for h in the statement of the theorem.



Now we have that:

$$\int_a^b \alpha(x)h(x)dx = \int_{x_0-\delta}^{x_0+\delta} \alpha(x)\left[\big(x - (x_0 - \delta)\big)\big((x_0 + \delta) - x\big)\right]^{n+1} dx$$

$$\geq \int_{x_0-\delta}^{x_0+\delta} \varepsilon\left[\big(x - (x_0 - \delta)\big)\big((x_0 + \delta) - x\big)\right]^{n+1} dx$$

$$= \varepsilon \int_{x_0-\delta}^{x_0+\delta} \left[\big(x - (x_0 - \delta)\big)\big((x_0 + \delta) - x\big)\right]^{n+1} dx > 0$$

(the last integral is > 0 because it is the integral of a positive function (except at the endpoints) over an interval of positive length). And so we get that:

$$\int_a^b \alpha(x)h(x)dx > 0.$$

But this is a contradiction since we said that the above integral must be zero for all such h. Thus $\alpha(x)$ must be zero on the whole interval $[a,b]$.

■

Some people might recognize the above lemma in its interpretation through inner products on a vector space of functions. In other words, Lemma 1.3.1 can be interpreted as follows. For any fixed non-negative integer n, consider the real vector space:

$$V = \{h \in C^n[a,b] : h(a) = 0 \quad and \quad h(b) = 0\}.$$

In other words, this is the vector space of n-times continuously differentiable functions on $[a,b]$ that vanish at a and b. Notice that this is a subspace of the bigger vector space of all continuous functions on $[a,b]$:

$$X = C^0[a,b]$$

coupled with the inner product:

$$\langle f,g\rangle = \int_a^b f(x)g(x)dx.$$

Then what Lemma 1.3.1 says is that if $\alpha \in X$ is a vector such that $\langle \alpha, h\rangle = 0$ for any vector $h \in V$, then α is equal to the zero vector. In other words, the only vector in X that is orthogonal to the subspace V is the zero vector. Symbolically this is written as $V^\perp = \{0\}$. This interpretation of the above lemma has an important significance in the calculus of variations and we will return to it at the end of the chapter in order to see how it presents an interesting analogy between the calculus of variations and multivariable calculus. However, if you didn't fully understand this interpretation then that's all right since we won't use it anywhere else in this book.

In multivariable calculus, we found local extrema of real-valued functions by first finding all of the points in their domains where their gradient is equal to zero (since a necessary condition for a point to be a local extremum of a real-valued function is that its gradient is zero at that point). Analogously, the following theorem presents a necessary condition for a curve $Y(x)$ to be a local extremum of a type of functional. The following necessary condition is called the "Euler-Lagrange differential equation" (note the two names attached to this equation!). It is our first major result in this book. Note that in the following theorem we give a necessary condition for local extremums. However, since all global extremums are local extremums the following theorem also acts as a necessary condition for global extremums as well.

Before we get to the proof of the following theorem, let's discuss the idea behind it. To do that, let us again rewind back to differential calculus and see how we found extremums of real-valued multivariable functions of the form $F : \mathbb{R}^2 \to \mathbb{R}$. Then we will try and see how we can take and apply these ideas to the case of functionals. Suppose that we know that the point $p = (x_0, y_0)$ is a local extremum of F. What condition must it satisfy? As you probably know, it must satisfy the condition that $\nabla F(x_0, y_0) = 0$ (0 on the right is the zero vector). How did we prove this in calculus class? Well, one way to prove this is to take any line $l(t) = \big(u(t), v(t)\big)$ that passes through our point (x_0, y_0) at time $t = t_0$. Then, since (x_0, y_0) is an extremum of F and $l(t)$ passes through this point at time $t = t_0$, we get that $t = t_0$ must be an extremum of the one variable function $F\big(l(t)\big) = F\big(x(t), y(t)\big)$ (this argument uses the continuity of $l(t)$). So we must have that:

$$\frac{d}{dt}\Big(F\big(x(t),y(t)\big)\Big)\Big|_{t=t_0} = 0.$$

Expanding the right-hand side gives is that:

$$\frac{d}{dt}\Big(F\big(x(t),y(t)\big)\Big)\Big|_{t=t_0} = \frac{\partial F}{\partial x}(x_0,y_0)\cdot x'(t_0) + \frac{\partial F}{\partial y}(x_0,y_0)\cdot y'(t_0)$$

$$= \nabla F(x_0,y_0)\cdot\big(x'(t_0),y'(t_0)\big) = 0$$

(· between two vectors here always mean "vector dot product"). And so we get that $\nabla F(x_0,y_0)$ is a vector that is perpendicular to $\big(x'(t_0),y'(t_0)\big)$. Since the line $l(t)$ can always be chosen such that $\big(x'(t_0),y'(t_0)\big)$ points in any direction, we get that the above equation implies that $\nabla F(x_0,y_0) = 0$ (since the only vector that is perpendicular to everything is the zero vector). Thus, if a point (x_0,y_0) if an extremum of F it must satisfy the condition that $\nabla F(x_0,y_0) = 0$.

The idea in the variational problem of finding the extremums of a functional is the same. If a curve $Y(x)$ is a local extremum of a functional, then if we take any linear flow $\Lambda(x,t)$ (analogous $l(t)$ above) that passes through our extremum at $t = t_0$, the one variable function being the functional composed with this linear flow should have an extremum at $t = t_0$ (we will use the continuity of Λ, or more precisely $\widetilde{\Lambda}$, to prove this assertion). Thus the time derivative of the functional composed with this linear flow should be zero at $t = t_0$. We will then apply the above lemma (which remember has an interpretation through an inner product) to this fact in order to derive a necessary condition that our extremum $Y(x)$ must satisfy. Let's begin.

Theorem 1.3.3 (The Basic Euler-Lagrange Differential Equation): *Let J be a functional defined by:*

$$J[y] = \int_a^b F(x,y,y')dx$$

where $F \in C^2[\mathbb{R}^3]$. Let J's domain be the set of curves $y(x) \in C^2[a,b]$ that satisfy the boundary conditions:

$$y(a) = A \quad and \quad y(b) = B$$

where A and B are two real numbers. Then a necessary condition for the curve $Y(x)$ to be a local extremum of the functional J is that it satisfies the Euler-Lagrange differential equation:

$$\frac{\partial F}{\partial y}\big(x,y(x),y'(x)\big) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\big(x,y(x),y'(x)\big)\right) = 0.$$

Here $\partial F/\partial y$ means the partial of F with respect to the argument that holds y (in our case the second argument of F) and $\partial F/\partial y'$ means the partial of F with respect to the argument that holds y' (in our case the third argument of F). This kind of "partial notation" may seem weird at first sight, but it's standard in the field and becomes completely user friendly as one uses it more *and more. If the reader is confused or would like to get more clarification on this "partial notation," I suggest going to the next example and see how the Euler-Lagrange differential equation is computed there and then come back to the proof of this theorem.*

If we are too lazy to write the arguments in the above differential equation (as I will often be), the Euler-Lagrange differential equation takes the shorter and nice-to-look-at form:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

Proof: There are many great proofs of this theorem. Our proof here will depend on the concept of smooth linear flows that we developed in the previous section.

Suppose that $Y(x)$ is an extremum our functional J. We want to prove that Y satisfies the Euler-Lagrange differential equation. Take any curve $h \in C^2[a,b]$ such that $h(a) = h(b) = 0$ and form the 2-smooth linear flow $\Lambda : [a,b] \times [-1,1] \to \mathbb{R}$ defined by:

$Equation$ 1.3.4: $$\Lambda(x,t) = Y(x) + h(x) \cdot t.$$

Notice that this flow Λ flows with the constant speed $h(x)$ (meaning $\frac{\partial \Lambda}{\partial t}(x,t) = h(x)$) and it passes through $Y(x)$ at time $t = 0$ (meaning $\Lambda(x,0) = Y(x)$). Now consider the real-valued function:

$$\mathcal{G}(t) = J[\Lambda(x,t)] = \int_a^b F\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right) dx$$

of the time variable t. Here what we did is for each time $t \in [-1,1]$ we plugged in the curve $\Lambda(x,t)$ into the argument of the functional J.

The first step in this proof is to realize that since the curve $\Lambda(x,0) = Y(x)$ is a local extremum of the functional J, $t = 0$ should be a local extremum of the function $\mathcal{G}(t) = J[\Lambda(x,t)]$. Thus, the derivative of $\mathcal{G}(t) = J[\Lambda(x,t)]$ should be zero at $t = 0$. Let's formally state and prove this observation.

Lemma 1.3.5: *Our function $\mathcal{G}(t)$ above is differentiable and its derivative at $t = 0$ is equal to zero:*

$$\mathcal{G}'(0) = \frac{d}{dt}\left(J[\Lambda(x,t)]\right)\Big|_{t=0} = \frac{d}{dt}\left(\int_a^b F\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right) dx\right)\Bigg|_{t=0} = 0.$$

Proof: The proof of this lemma is not at all hard. We have that the function $\mathcal{G}(t)$ is differentiable because we can carry the time derivative under the integral sign (since the above integrand is continuously differentiable and our domain of integration is compact). Now in order to prove that the above derivative is equal to zero, it is sufficient to just show that $t = 0$ is a local extremum of our function $\mathcal{G}(t)$. The fact that $\mathcal{G}'(0) = 0$ will then follow from the well-known fact that the derivative of a function at an extremum is equal to zero.

Let's suppose that $Y(x)$ is a local minimum of the functional J. The proof of this lemma in the case of when $Y(x)$ is a local maximum of the functional J is similar. Then by definition there exists a $\delta > 0$ such that for any $h \in \text{dom}(J)$ such that $\|h - Y\| \leq \delta$,

$$J[h(x)] \geq J[Y(x)].$$

Now, by Theorem 1.2.12 we have that our 2-smooth linear flow $\Lambda(x,t)$ flows continuously through the space of curves $\text{dom}(J) \subseteq C^2[a,b]$. So there exists a $\Delta > 0$ such that for any time $t \in [-1,1]$ such that $|t - 0| \leq \Delta$,

$$\|\Lambda(x,t) - \Lambda(x,0)\| = \|\Lambda(x,t) - Y(x)\| \leq \delta.$$

This means that for any $t \in [-\Delta, \Delta]$,

$$J[\Lambda(x,t)] \geq J[Y(x)] = J[\Lambda(x,0)].$$

Or in other words: for any $t \in [-\Delta, \Delta]$,

$$\mathcal{G}(t) \geq \mathcal{G}(0).$$

So $t = 0$ is a local extremum of $\mathcal{G}(t)$ and thus $\mathcal{G}'(0) = 0$.

■

The above lemma is key because it allows us to relate the necessary condition for the functional J to have an extremum at the curve $Y(x)$ to the necessary condition for the function $J[\Lambda(x,t)]$ of the one variable t to have an extremum at the point in time when Λ passes through $Y(x)$ (the time $t = 0$ that is).

Now we are ready to do the exciting calculation that proves the theorem. In the previous lemma we proved that for our linear flow $\Lambda(x,t)$:

$$\frac{d}{dt}\left(J[\Lambda(x,t)]\right)\Big|_{t=0} = \frac{d}{dt}\left(\int_a^b F\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right) dx\right)\Bigg|_{t=0} = 0.$$

Carrying the derivative under the integral sign gives us that:

$$\frac{d}{dt}\left(J[\Lambda(x,t)]\right)\Big|_{t=0} = \int_a^b \frac{\partial}{\partial t}\left(F\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\right)\Bigg|_{t=0} dx$$

$$= \int_a^b \left(\frac{\partial F}{\partial y}\left(x, \Lambda(x,0), \frac{\partial \Lambda(x,0)}{\partial x}\right) \cdot \frac{\partial \Lambda}{\partial t}(x,0) + \frac{\partial F}{\partial y'}\left(x, \Lambda(x,0), \frac{\partial \Lambda(x,0)}{\partial x}\right) \cdot \frac{\partial^2 \Lambda}{\partial x \partial t}(x,0)\right) dx = 0.$$

Since $\Lambda(x,0)$ is equal to $Y(x)$ (see Equation 1.3.4), we get that the above equation can be rewritten as:

$$\int_a^b \left(\frac{\partial F}{\partial y}(x,Y,Y')\cdot\frac{\partial \Lambda}{\partial t}(x,0)+\frac{\partial F}{\partial y'}(x,Y,Y')\cdot\frac{\partial^2 \Lambda}{\partial x\partial t}(x,0)\right)dx=0.$$

Since $\frac{\partial^{k+1}\Lambda}{\partial x^k\partial t}(x,0)=h^{(k)}(x)$ for $\forall k\in\{0,1\}$ (see Equation 1.3.4), we then get that the above equation can further be rewritten as:

$$\int_a^b \left(\frac{\partial F}{\partial y}\cdot h(x)+\frac{\partial F}{\partial y'}\cdot h'(x)\right)dx=0$$

(here I'm too lazy to write – and it makes the equations look shorter – the argument of $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial y'}$. As before they're being evaluated at (x,Y,Y')). Integrating the term $\int_a^b \frac{\partial F}{\partial y'}h'(x)dx$ by parts gives that (remember $h(a)=h(b)=0$):

$$\int_a^b \frac{\partial F}{\partial y'}h'(x)dx=\frac{\partial F}{\partial y'}\cdot h(x)\Big|_{x=a}^{x=b}-\int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)h(x)dx=-\int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)h(x)dx.$$

With this we get that the equation before becomes:

$$\int_a^b \frac{\partial F}{\partial y}h(x)dx-\int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)h(x)dx=\int_a^b \left(\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)h(x)dx=0,$$

and so:

$$\int_a^b \left(\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)h(x)dx=0.$$

To be explicit, let's rewrite the above equation with the argument $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial y'}$ included:

$$\int_a^b \left(\frac{\partial F}{\partial y}(x,Y,Y')-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}(x,Y,Y')\right)\right)h(x)dx=0.$$

Let's see what we have proven. We have proven that for any $h\in C^2[a,b]$ such that $h(a)=h(b)=0$, the above integral holds (since the h that we chose to define Λ was arbitrary from this set). So, we can apply Lemma 1.3.1 to the above equation to finally get that:

$$\frac{\partial F}{\partial y}(x,Y,Y')-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}(x,Y,Y')\right)=0$$

on the whole interval $[a,b]$. But this is the extremum curve $Y(x)$ being plugged into the Euler-Lagrange differential equation! So, we get that our extremum curve $Y(x)$ does indeed satisfy the

Euler-Lagrange differential equation. With this we have proved that a necessary condition for $Y(x)$ to be an extremum of the functional J is that it satisfies the Euler-Lagrange differential equation. With this we have proven the theorem.

■

Amazing! The following is a very cool definition to make right after proving the above theorem:

Definition 1.3.6: *Let J be a functional defined by:*

$$J[y] = \int_a^b F(x, y, y')dx$$

where $F \in C^2[\mathbb{R}^3]$. Let J's domain be the set of curves $y(x) \in C^2[a, b]$ that satisfy the boundary conditions:

$$y(a) = A \quad and \quad y(b) = B$$

where A and B are two real numbers. Then the ***variational derivative****, or* ***functional derivative****, of the functional J is defined as:*

$$\frac{\delta J}{\delta y}[y] = \frac{\partial F}{\partial y}(x, y, y') - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}(x, y, y')\right).$$

This is the left-hand side of the Euler-Lagrange differential equation. Notice that the necessary condition proved in the above theorem can now be rewritten in the nice form that a necessary condition for a curve $Y(x)$ to be an extremum of J is that it satisfies:

$$\frac{\delta J}{\delta y}[Y] \equiv 0.$$

In other words, the variational derivative of the functional J must be equal to zero at the curve $Y(x)$. This equation has the benefit that it looks like the analog necessary condition for a real-valued function of one variable to have an extremum at a point: that its derivative is equal to zero at that point. We will later see however that multivariable calculus provides a much better analogy to the calculus of variations rather than real-valued one variable calculus.

Notice the interesting feature of the variational derivative $\frac{\delta J}{\delta y}$ in that it is not a functional. In fact, it takes a curve $y(x)$ and outputs another curve/function of x, namely the left-hand side of the Euler-Lagrange differential equation. As we'll later discuss at the end of the chapter, the variational derivative is in fact analogous to the gradient function for a real-valued multivariable function. The variational derivative will play a crucial role in later chapters.

Using the calculus of variations to solve problems is exciting and I would like to show you some examples of how the calculus of variations is used to solve some classical problems in mathematics and physics. Up to this point we have developed a theory of necessary conditions for extremums of functionals (Theorem 1.3.3). However, we haven't developed any kind of theory of sufficient conditions for curves to be extremums of our functionals. For example, technically for any functional of the form:

$$J[y] = \int_a^b F(x, y, y')dx,$$

by looking at the solutions to its Euler-Lagrange differential equation we can only tell which curves have the potential to be extremums of J (since the Euler-Lagrange differential equation is only a necessary condition for a curve to be an extremum for the functional above). However, at our current stage we yet have no way to actually prove that the curves that we find are actually extremums of the functional J. The theory that allows one to prove that a curve is an extremum of a functional is called the theory of the second variation (kind of like the second derivative of a function), and we will not develop this theory for some time.[4]

However, in many cases, such as in the following examples, it is physically obvious (or "intuitively clear") that a functional has one and only one extremum. In those cases, what one often does in order to find the extremum of their functional is they find the solution to the functional's Euler-Lagrange differential equation and then claim that that solution is the extremum that they are looking for. Of course, that's not really rigorous because the last claim needs proof. But for those who just seek answers and are comfortable with a little bit of loose rigor, this is just fine. So, for the purpose of showing the power of the Euler-Lagrange differential equation in its ability to find extremums, in the following examples I will temporarily assume this attitude in that I will use my (our?) physical intuition to assume that the functional in question has one and only one global extremum and that it is the solution to the Euler-Lagrange differential equation. But the reader should not be discouraged because when we do reach the chapter on the theory of the second variation of a functional, we will return to every single of the following examples to prove that the extremum that we found was in fact the extremum of the functional in question.

An analog of the above attitude in ordinary differential calculus would be to look for extremums of functions by only looking at their first derivatives and not their second. But again, we will return to every one of the following examples after we study a bit of second variational theory in order to rigorously prove that the solutions that we find are actually the extremums of the functionals in question. Let's begin.

Example 1.3.7: Let us take the functional:

$$J[y] = \int_a^b \sqrt{1 + y'^2}dx = \int_a^b F(x, y, y')$$

defined on the set of curves $y(x) \in C^2[a, b]$ that satisfy the boundary conditions:

$$y(a) = A \quad and \quad y(b) = B$$

[4] For second variation theory, look at future editions of this book.

where A and B are two real numbers. Let us find this functional's minimum. Notice that the above integral is the arc-length integral. This J is in fact the arc-length functional and so as mentioned before, the problem of finding the minimum of the above functional is equivalent to finding the curve $y(x)$ that connects the two points (a, A) and (b, B) and that has the minimum arc-length. The Euler-Lagrange differential equation of this functional is:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial}{\partial y}\left(\sqrt{1+y'^2}\right) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(\sqrt{1+y'^2}\right)\right).$$

Calculating the partials $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial y'}$ is like treating y and y' as one would variables and then taking partials. So $\frac{\partial}{\partial y}\left(\sqrt{1+y'^2}\right) = 0$ because there is no y in $\sqrt{1+y'^2}$ and $\frac{\partial}{\partial y'}\left(\sqrt{1+y'^2}\right) = \frac{y'}{\sqrt{1+y'^2}}$. So we get that the Euler-Lagrange differential equation of this functional is:

$$\frac{\partial}{\partial y}\left(\sqrt{1+y'^2}\right) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(\sqrt{1+y'^2}\right)\right) = -\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0.$$

Multiply both sides by -1 to get that:

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0.$$

In order to find the minimum of our functional we need to find the solution of this second order differential equation that satisfies the boundary conditions $y(a) = A$ and $y(b) = B$ (it needs to satisfy these boundary conditions in order to be in the domain of J). Integrating both sides of the above equation gives:

$$\frac{y'}{\sqrt{1+y'^2}} = c$$

for some constant of integration c. Multiply through by $\sqrt{1+y'^2}$ and then square both sides to get that:

$$y'^2 = c^2\left(1+y'^2\right).$$

From this equation it is easy to see that $c^2 = 1$ is <u>impossible</u>. So, we can solve for y'^2 to get that:

$$y'^2 = \frac{c^2}{1-c^2}.$$

This equation implies that y' is constant on all of $[a, b]$. So we get that y must be a line:

$$y(x) = \alpha x + \beta,$$

where α and β are constants. Great! Now, notice that every of the above calculus and algebraic step was reversible.[5] This implies that lines are and are the only solutions to our Euler-Lagrange differential equation. So our extremum curve must be the line that connects the two points (a, A) and (b, B). Let's find the equation for this line. For this, we need to find α and β such that:

$$\alpha a + \beta = A,$$

$$\alpha b + \beta = B.$$

Solving this system of equations gives us that:

$$\alpha = \frac{dA - bB}{ad - bc},$$

$$\beta = \frac{aB - cA}{ad - bc}.$$

And so, the extremum of our functional is:

$$y(x) = \frac{dA - bB}{ad - bc} x + \frac{aB - cA}{ad - bc}.$$

With this we have found that the curve that connects the two points (a, A) and (b, B) and that has the minimum arc-length (or in other words that minimizes our arc-length functional J) is the straight line that connects the two points (a, A) and (b, B), the equation for which is given right above. The answer that we got that the shortest path between two points is a line shouldn't be at all be surprising to the reader. However, it's exciting to see how the Calculus of Variations can give us a proof of this very familiar result.[6]

For the next example, let us prove a small but very useful identity that often helps make calculations much shorter.

Theorem 1.3.8 (Beltrami's Identity): *Let J be a functional defined by:*

$$J[y] = \int_a^b F(x, y, y') dx$$

[5] A reader with a good eye might be worried that the step where we squared both sides of the equation is not a reversible step. It is in fact reversible because notice that there stands an arbitrary constant. Upon doing the reverse step of square rooting in that step, any possible negative signs that might pop up will be engulfed by the arbitrary constant c (meaning c might become $-c$ upon square rooting, but since it's an arbitrary constant we can always relabel it to be c again).

[6] Again, we will come back to this example after we talk about the second variation to actually prove that this is the minimum of our functional [see future edition of this book].

where $F \in C^2[\mathbb{R}^3]$. Let J's domain be the set of curves $y(x) \in C^2[a, b]$ that satisfy the boundary conditions:

$$y(a) = A \quad and \quad y(b) = B$$

where A and B are two real numbers. Suppose also that the integrand F does not explicitly depend on x (like for example $F = \sqrt{1 + y'^2}$, $F = y^2 + y'^2$, and $F = y^2 \sin(y')$ are all examples of where F does not explicitly depend on x. Another way to say that "F does not explicitly depend on x" is that $\frac{\partial f}{\partial x}$ is zero everywhere).

a.) Then the solutions of the Euler-Lagrange differential equation are also solutions of the differential equation:

$$Equation\ 1.3.9 \qquad F(x, y, y') - y' \frac{\partial F}{\partial y'}(x, y, y') = C,$$

where C is some constant.[7] *Unfortunately the set of solutions of this differential equation is not necessarily equal to that of the Euler-Lagrange differential equation. But it almost always is, and part (b) gives an easy-to-use condition for checking when the two set of solutions are equal.*

b.) If it so happens that for any solution y to Equation 1.3.9, y' vanishes only on a discrete set, then this differential equation is equivalent to the Euler-Lagrange differential equations (meaning that they have the same exact solutions).

Proof: Let's first prove part a. Take the Euler-Lagrange differential equation of J:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

Calculating the $\frac{d}{dx}$ on the left-hand side gives that the above equation becomes:

$$\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial y' \partial y} y' - \frac{\partial^2 F}{\partial y'^2} y'' = 0.$$

Now Beltrami must have been really bothered by the $\frac{\partial F}{\partial y}$ term, and so he decided to integrate it. In order to integrate it in x we need to multiply it by y' and then add a $\frac{\partial F}{\partial y'} y''$ term to it (so as to get this term to be the derivative of F, which is what is meant by "integrating it"). So, take the above equation and multiply it through by y' and then add $\frac{\partial F}{\partial y'} y'' - \frac{\partial F}{\partial y'} y''$ (which is zero) to the left side. With this we get that the above equation becomes:

[7] I would call this the integrated form of the Euler-Lagrange differential equation (the reason for this will be seen in the proof of this theorem). The purpose of this differential equation is that in the case when the integrand F does not explicitly depend on x, this differential equation is sometimes easier to handle rather than the original Euler-Lagrange differential equation of the functional.

$$\frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' - \frac{\partial F}{\partial y'}y'' - \frac{\partial^2 F}{\partial y' \partial y}y'^2 - \frac{\partial^2 F}{\partial y'^2}y''y' = 0.$$

Now we can integrate the term $\frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y''$ term. Since $\frac{d}{dx}(F) = \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y''$ (as planned), if we take the antiderivative of both the left and right hand sides of the above equation, we get that:

$$\int \left(\frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' - \frac{\partial F}{\partial y'}y'' - \frac{\partial^2 F}{\partial y' \partial y}y'^2 - \frac{\partial^2 F}{\partial y'^2}y''y'\right)dx = \int 0dx$$

is given by:

$$\text{Equation } 1.3.10 \qquad F - \int \left(\frac{\partial F}{\partial y'}y'' + \frac{\partial^2 F}{\partial y' \partial y}y'^2 + \frac{\partial^2 F}{\partial y'^2}y''y'\right)dx = C$$

where C is some arbitrary constant of integration. We don't like having integrals in differential equations. So what can we do? Let us try integrating the term $\frac{\partial F}{\partial y'}y''$ by parts:

$$\int \frac{\partial F}{\partial y'}y''dx = \frac{\partial F}{\partial y'}y' - \int \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)y'dx = \frac{\partial F}{\partial y'}y' - \int \left(\frac{\partial^2 F}{\partial y' \partial y}y'^2 + \frac{\partial^2 F}{\partial y'^2}y''y'\right)y'dx,$$

and so:

$$\int \frac{\partial F}{\partial y'}y''dx = \frac{\partial F}{\partial y'}y' - \int \left(\frac{\partial^2 F}{\partial y' \partial y}y'^2 + \frac{\partial^2 F}{\partial y'^2}y''y'\right)y'dx.$$

Plugging this into Equation 1.3.10 gives that:

$$F - \frac{\partial F}{\partial y'}y' + \int \left(\frac{\partial^2 F}{\partial y' \partial y}y'^2 + \frac{\partial^2 F}{\partial y'^2}y''y'\right)y'dx - \int \left(\frac{\partial^2 F}{\partial y' \partial y}y'^2 + \frac{\partial^2 F}{\partial y'^2}y''y'\right)dx = C.$$

The two integrals cancel out and so we get that:

$$F - \frac{\partial F}{\partial y'}y' = C,$$

which is the differential equation that we wanted to get. With these algebra and calculus steps we have shown that any solution of the Euler-Lagrange differential equation is also a solution of the above differential equation. With this we have proved part (a). Now let's prove part (b).

The reason why we need such a tricky condition for the two differential equations to be equivalent is that there was one (and only one) non-reversible step that we did above when we went from the Euler-Lagrange differential equation to the differential equation $F - \frac{\partial F}{\partial y'}y' = C$. That was the step when we multiplied the equation:

$$\text{Equation } 1.3.11 \qquad \frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial y' \partial y}y' - \frac{\partial^2 F}{\partial y'^2}y'' = 0$$

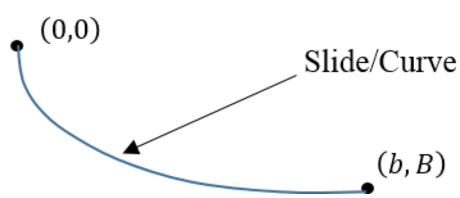
through by y' to get:

$$\text{Equation 1.3.12} \qquad \frac{\partial F}{\partial y}y' - \frac{\partial^2 F}{\partial y'\partial y}y'^2 - \frac{\partial^2 F}{\partial y'^2}y''y' = 0$$

(this wasn't written out as a separate step, but was indicated by the phrase "…So, take the above equation and multiply it through by y' and then…"). This is not a reversible step since if Equation 1.3.12 holds, Equation 1.3.11 doesn't necessarily hold because y' might be the quantity that is zero in equation 1.3.12. But in the case that y' is zero only on a discrete set, then by the continuity of the quantity $\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial y'\partial y}y' - \frac{\partial^2 F}{\partial y'^2}y''$ it is easy to see that Equation 1.3.12 does imply that Equation 1.3.11 holds. With this we get that the condition that y' is zero only on a discrete set implies that all of the above steps are reversible. So if this "y' is zero only on a discrete set" condition holds on every solution of the differential equation $F - \frac{\partial F}{\partial y'}y' = C$, working all of those steps backwards we see that every solution of this differential equation is also a solution of the Euler-Lagrange differential equation and thus these two differential equations are equivalent. With this we have proved the theorem.

■

Example 1.3.13: Let us solve the Brachistochrone problem (see example 1.1.8). Let us say that we want to build a slide that starts at $(0, 0)$ and goes down to (b, B) where $b > 0$ and $B < 0$ and that has the minimum descent time for a sliding point particle (disregarding friction) that starts at rest at $(0, 0)$[8]. What should the shape of the slide be?



Let's figure out the integral expression for the time of descent for a point particle to slide down from the point $(0, 0)$ to the point (b, B) along a curve $y(x)$ that connects these two points. Let's assume that the curves $y(x)$ for the slide that we are considering all have non-positive derivative (so as to guarantee that the point particle can even slide down from the point $(0, 0)$ to the point (b, B)). Let's take any such curve $y(x)$. By the conservation of energy, we know that (remember, the point particle starts at rest at $(0, 0)$):

$$\frac{1}{2}mv^2 = -mgy$$

where m is the mass of the particle, v is its total speed, g is the gravitational constant 9.8 $meters/second^2$. Solving for v we get the famous equation:

[8] Point particle means that all of its mass is centered at some point, which is sliding down the curve.

$$v = \sqrt{-2gy}.$$

Don't worry, the above quantity is real and never imaginary since $y(x) \leq 0$ (because $y(0) = 0$ and we agreed that y has non-positive derivative). Now partition up the interval $[0, b]$ into many many small subintervals of equal length. For any small partition interval $[x_k, x_{k+1}]$, the descent time for the point particle to slide down the small piece of curve over this small interval is approximately:

$$\frac{\sqrt{(x_{k+1} - x_k)^2 + \left(y(x_{k+1}) - y(x_k)\right)^2}}{\sqrt{-2gy(x_k)}} = \frac{\sqrt{1 + \left(\frac{y(x_{k+1}) - y(x_k)}{x_{k+1} - x_k}\right)^2}}{\sqrt{-2gy}}(x_{k+1} - x_k)$$

$$\approx \frac{\sqrt{1 + \left(y'(x_k)\right)^2}}{\sqrt{-2gy(x_k)}}\Delta x$$

where $\Delta x = x_{k+1} - x_k$ (remember, all of the partition intervals have equal length). Summing this up over all of the partition intervals and then passing over to the limit as the length of each of the partition intervals becomes smaller and smaller gives that the descent time of the point particle over the whole curve is given by the integral:

$$\int_0^b \frac{\sqrt{1 + \left(y'(x)\right)^2}}{\sqrt{-2gy(x)}}dx = \int_0^b \frac{1}{\sqrt{2g}}\frac{\sqrt{1 + y'^2}}{\sqrt{-y}}dx.$$

So now we can reformulate the problem of finding the curve of minimal descent time from the point $(0, 0)$ to the point (b, B) as finding the minimum of the functional:

$$J[y] = \int_0^b \frac{1}{\sqrt{2g}}\frac{\sqrt{1 + y'^2}}{\sqrt{-y}}dx = \int_0^b F(x, y, y')dx,$$

whose domain is the set of curves y that satisfy the boundary conditions:

$$y(0) = 0 \quad and \quad y(b) = B.$$

Let us find this functionals minimum by finding the solution of its Euler-Lagrange differential equation. Since the integrand does not depend on x explicitly, we can apply Beltrami's identity to get that any solution of this functional's Euler-Lagrange differential equation must also satisfy the differential equation:

$$F - y'\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{2g}}\frac{\sqrt{1 + y'^2}}{\sqrt{-y}} - y'\frac{\partial}{\partial y'}\left(\frac{1}{\sqrt{2g}}\frac{\sqrt{1 + y'^2}}{\sqrt{-y}}\right) = C$$

for some constant C. We will soon see that all solutions of this differential equation satisfy the condition in part (b) of Beltrami's Identity (Theorem 1.3.8). So by solving this differential equation we will have solved the Euler-Lagrange differential equation as well. Multiply through the above equation by $\sqrt{2g}$ to get that:

$$\frac{\sqrt{1+y'^2}}{\sqrt{-y}} - y' \frac{\partial}{\partial y'}\left(\frac{\sqrt{1+y'^2}}{\sqrt{-y}}\right) = \sqrt{2g}C.$$

Since C is an arbitrary constant, $\sqrt{2g}C$ is also an arbitrary constant and so we can relabel $\sqrt{2g}C$ to be C again:

$$\frac{\sqrt{1+y'^2}}{\sqrt{-y}} - y' \frac{\partial}{\partial y'}\left(\frac{\sqrt{1+y'^2}}{\sqrt{-y}}\right) = C.$$

Calculating the $\frac{\partial}{\partial y'}$ above gives us that the above equation becomes:

$$\frac{\sqrt{1+y'^2}}{\sqrt{-y}} - \frac{y'^2}{\sqrt{1+y'^2}\sqrt{-y}} = C.$$

Multiplying through this equation by $\sqrt{1+y'^2}\sqrt{-y}$ and then squaring both sides gives that:

$$1 = -C^2(1+y'^2)y.$$

Relabeling $-C^2$ by C (we can always do this since these are arbitrary constants) and then dividing through the above equation by $(1+y'^2)y$ gives us that our solution must satisfy the differential equation:

$$\frac{1}{(1+y'^2)y} = C$$

for some constant C. Honestly, I don't know how people solved this differential equation. I guess Bernoulli sat down one day and decided to try plugging in all of the types of curves he knew into the above differential equation. He probably tried the circle and some other curves. Physics says that the most efficient curve in terms of descent time must have a vertical derivative at $(0,0)$ so that the particle can gain speed in the beginning.[9] Bernoulli must have first tried those curves. In

[9] An insight due to my father: Igor Grebnev.

any case, he finally stumbled onto the fact that the class of curves that solve the above differential equation are the cycloids, which have the parametric form:

$$x(t) = r(t - \sin(t)) + s$$

$$y(t) = r(-1 + \cos(t))$$

where r and s are constants. Cycloids represent the curve that a pebble takes when it is attached to the edge of a wheel when it's rolling. Let's check that the cycloids indeed satisfy the above differential equation. We have that:

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-r\sin(t)}{r(1-\cos(t))} = \frac{-\sin(t)}{1-\cos(t)}.$$

And so, plugging this into the left-hand side of the above differential equation gives:

$$\frac{1}{\left(1+\left(\frac{dy}{dx}\right)^2\right)y} = \frac{1}{\left(1+\left(\frac{y'(t)}{x'(t)}\right)^2\right)y} = \frac{1}{\left(1+\left(\frac{-\sin(t)}{1-\cos(t)}\right)^2\right)r(-1+\cos(t))}.$$

Simplifying this further gives that the above quantity becomes:

$$\frac{-1}{r}\frac{1}{\frac{(1-\cos(t))^2+\sin^2(t)}{1-\cos(t)}} = \frac{-1}{r}\frac{1}{\frac{2-2\cos(t)}{1-\cos(t)}} = -\frac{1}{2r},$$

which is indeed a constant (meaning here $C = \frac{-1}{2r}$). With this we have checked that the cycloids are indeed solutions of the differential equation $\frac{1}{\left(1+y'^2\right)y} = C$. In fact, using the uniqueness theorems from differential equation theory it is easy to check that the cycloids are the only solutions to this differential equation.

Notice that every step above getting from $F - y'\frac{\partial F}{\partial y'} = C$ to the differential equation $\frac{1}{\left(1+y'^2\right)y} = C$ was reversible.[10] So working all of those steps backwards we see that the solutions and only solutions to the differential equation $F - y'\frac{\partial F}{\partial y'} = C$ are the cycloids. Now, since the derivative of cycloids are zero on isolated sets (easy to check), we get by part (b) of Beltrami's Identity that the cycloids are also solutions and the only solutions of the Euler-Lagrange differential equation of our functional. So to find the minimum of our functional we just need to find the cycloid that passes through our two points $(0,0)$ and (b,B). For specific values of b and B this can always be done using computers.

With this we get that the minimum descent time curve from the point $(0,0)$ to (b,B) is the cycloid that passes through these two points. We have solved the Brachistochrone problem. Solving this problem without using Beltrami's identity by directly looking at the Euler-Lagrange

[10] See footnote 5 on page 23. It applies here as well.

differential equation is computationally a much more formidable task. You can try writing out the Euler-Lagrange differential equation for this functional to see what I mean.

I remember that when our first-year calculus professor posed the Brachistochrone problem to our class, I immediately became interested in the problem. That same day I went on Wikipedia (the great source of mathematical knowledge) to learn more about this problem and the legend surrounding it. The Wikipedia page on this problem indicated that there are several solutions to this problem (all of which were beyond my reach at the time), one of which is a direct application of the Euler-Lagrange differential equation. All solutions are variational in nature and have one or another connection the calculus of variations. In the exercises I wrote up a problem that suggests a different and very ingenuous approach to this problem, one that use the concept of light [see future edition of this book].

Section 4: Concluding Words

As I mentioned several times, multivariable calculus provides an excellent analog to the calculus of variations. In fact, the calculus of variations can be thought of as an infinite dimensional extension of multivariable calculus. Let J be a functional of the form:

$$J[y] = \int_a^b F(x, y, y')dx$$

defined on some set $E = \{h \in C^2[a, b] : h(a) = h(b) = 0\}$ and let $G : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a real-valued multivariable function. Here are the analogies between the two:

1. The domains of both J and G are subsets of some vector space.
2. As will be explained in point 4, the variational derivative $\frac{\delta J}{\delta y}$ is the analog of ∇G. Notice that the type of elements in the domain and range of both the variational derivative and gradient are the same. In other words, the variational derivative maps curves to curves while the gradient maps vectors to vectors.
3. In the proof of the Euler-Lagrange differential equation, we proved in Lemma 1.3.5 that a necessary condition for the functional J to have an extremum at a curve Y is that for any n-smooth linear flow $\Lambda(x, t)$ that passes through Y, the function $J[\Lambda(x, t)]$ must have an extremum at the point in time when Λ passes through Y. The analog necessary condition for G to have an extremum at a point x is that for any line $l(t)$ (or "linear function") that passes through x, the one variable function:
$$G\big(l(t)\big)$$
must have an extremum at the point in time when l passes through the point x. $l(t)$ is the analog of Λ since as was mentioned right after Definition 1.2.8, n-smooth linear flows such as Λ can be called the "lines" of the space $C^k[a, b]$ and even E.
The next point explains the analog between how these conditions imply that $\frac{\delta J}{\delta y} = 0$ and $\nabla G = 0$ at their respective extrema.

4. As mentioned after Lemma 1.3.1, Lemma 1.3.1 has an interpretation through an inner product of the form:

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx = 0.$$

The interpretation is that if $\alpha(x)$ was a function such that $\langle \alpha, h\rangle = 0$ for every $h \in C^2[a,b]$ that satisfied the boundary conditions $h(a) = h(b) = 0$, then $\alpha \equiv 0$. Near the end of the proof of the Euler-Lagrange differential equation (Theorem 1.3.3) we applied Lemma 1.3.1 to say that:[11]

$$\int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) h(x)dx = 0,$$

which notice can also be written as:

$$\left\langle \frac{\delta J}{\delta y}[Y], h\right\rangle = 0$$

implies that:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\delta J}{\delta y}[Y] \equiv 0$$

Now let's see where the analog is with the multivariable function G. The analog is that a necessary condition for G to have an extremum at a point x is that for any vector h, the dot product (which is an inner product):

$$\nabla G(x) \cdot h = 0.$$

An analog of the application of Lemma 1.3.1 to the functional J is to conclude from here that this implies that:

$$\nabla G = 0.$$

This is indeed a necessary condition for G to have an extremum at x and it is the analog condition for the variational derivative $\frac{\delta J}{\delta y}$ to be equal to the zero function (which is the zero vector in the vector space $E = \{h \in C^k[a,b] : h(a) = h(b) = 0\}$).

5. Another way to see the analog is to view a function $h \in E \subseteq C^k[a,b]$ as a vector with infinite coordinates with values in each coordinate being $h(x)$ for every $x \in [a,b]$. While a vector in $U \subseteq \mathbb{R}^n$ is a vector with n coordinates. For those who know a little bit more set theory, the analogy meant here is the analogy between the equations (here $\prod$ is the cartesian product):

$$\mathrm{dom}(J) \subseteq \prod_{\phi \in [a,b]} \mathbb{R}$$

$$\mathrm{dom}(G) \subseteq \prod_{\phi \in \{1,2,\ldots,n\}} \mathbb{R}$$

[11] Here the partials of F are being evaluated at (x, Y, Y') (see the end of the proof of Theorem 1.3.3).

Chapter 2: Generalizations of the Euler-Lagrange Differential Equation

"– I feel that to do well on the [AP] Calculus BC Test, you just have to memorize a lot of formulas" – Person A

"– Yeah" – Person B (author's friends)

Section 1: Overview (Very Short)

In this chapter we'll develop some more general forms of the Euler-Lagrange equation. The first of these generalized forms deals with the question of how to find extremums of functionals that take multiple curves for their arguments. Following that we will derive the Euler-Lagrange Differential Equation for functionals whose arguments are not curves, but surfaces or hypersurfaces.

Section 2: Multivariable Functionals

There are instances in mathematics (as we will encounter) when a functional is dependent on multiple functions of one variable. An example of such a functional is the functional that gives the arclength of a parametrized curve in $\mathbb{R}^2$:

$$J[u(t), v(t)] = \int_{t_0}^{t_1} \sqrt{\left(u'(t)\right)^2 + \left(v'(t)\right)^2}\,dt.$$

Here the functional takes two continuously differentiable functions $u, v : [t_0, t_1] \to \mathbb{R}$ and returns the arclength of the parametrized curve $\big(u(t), v(t)\big)$ that sits in $\mathbb{R}^2$. As we can see, this is a functional that takes two functions of one variable $u(t)$ and $v(t)$, and outputs a real number. This kind of functional that takes more than one of a type of argument (in this case functions of one variable) is naturally called a **multivariable functional**. So, we can of course ask the question of how one proceeds to find the extremums of such multivariable functionals? Let's investigate this question. In this section we will only treat the case of multivariable functionals of two variables (meaning functionals of the form $J[u(t), v(t)]$). The generalization of all the results that we will get to the case of functionals with even more variables will become trivial.

This kind of transition from one variable functional theory (which we were doing in the previous chapter) to multivariable functional theory is analogous to the transition that one does when one goes from learning one variable calculus to multivariable calculus (a milestone in a mathematician's life). So in order for us to make this transition from one variable functional theory to multivariable functional theory, let's review how we did this transition in normal differential calculus.

Suppose that we have a very nice (smooth) function $F(x, y)$. How did we find its local extremums? Well, the first thing we realized was that if a point (a, b) is a local extremum of $F(x, y)$, then if we fix the x component to be equal to a and let y be variable we should get that $y = b$ is a local extremum of the one variable function $F(a, y)$ of the variable y. What is a necessary condition for $y = b$ to be a local extremum of the function $F(a, y)$? A necessary condition is that the derivative of the function $F(a, y)$ at $y = b$ is zero:

$$\frac{d}{dy}\big(F(a, y)\big)\Big|_{y=b} = \frac{\partial F}{\partial y}(a, b) = 0.$$

If we do the same thing but this time fixing $y = b$, we get that $x = a$ must be a local extremum of the one variable function $F(x, b)$ and thus:

$$\frac{d}{dx}\big(F(x, b)\big)\Big|_{x=a} = \frac{\partial F}{\partial x}(a, b) = 0.$$

So, we got that a necessary condition for a point (a, b) to be an extremum of the multivariable function $F(x, y)$ is that the following two equations hold:

$$\frac{\partial F}{\partial x}(a, b) = 0,$$

$$\frac{\partial F}{\partial y}(a, b) = 0.$$

We often write this in the more compact form:

$$\nabla F(a, b) = 0$$

The idea behind the analog transition in the calculus of variations from one variable to multiple variables is exactly the same. It turns out that a necessary condition for a multivariable functional to have an extremum somewhere is that it satisfies the necessary condition for an extremum in each variable, just like with the F above. But before we can proceed to prove this, we have to define some sense of distance in the domain of multivariable functionals. For this, we make a couple of definitions.

In order to make the statements of some of our definitions and theorems precise and a bit shorter, let's agree to the following (quite standard) notation:

Notation 2.2.1: Let $\prod_{k=1}^{n} C^n[a,b]$ denote the set of all vectors of the form $\big(u_1(x), u_2(x), \dots, u_n(x)\big)$ where each $u_k(x)$ is a function in $C^n[a,b]$. Colloquially we might call elements of $\prod_{k=1}^{n} C^n[a,b]$ "**function vectors**" because they are vectors of functions.

Definition 2.2.2: *Let $\big(u_1(x), u_2(x), \dots, u_n(x)\big) \in \prod_{k=1}^{n} C^n[a,b]$ be a vector of functions each of which is in $C^n[a,b]$. Then, let's define the norm of this function vector by:*

$$\|(u_1, u_2, \dots, u_n)\| = \max_{k\in\{1,2,\dots,n\}}\{\|u_k\|\}.$$

Where $\|u_k(x)\|$ here denotes the norm that we defined on $C^n[a,b]$ (see definition 1.2.3). Again, it is easy to check that this definition of norm satisfies the four axioms of a normed space.

One with a bit of experience in abstract linear algebra might realize that since $\prod_{k=1}^{n} C^n[a,b]$ is a vector space, the above norm on vectors in $\prod_{k=1}^{n} C^n[a,b]$ naturally induces the following definition of distance between two "function vectors:"

Definition 2.2.3: *Let $\big(u_1(x), u_2(x), \dots, u_n(x)\big)$ and $\big(v_1(x), v_2(x), \dots, v_n(x)\big)$ be two function vectors in $\prod_{k=1}^{n} C^n[a,b]$. Then the distance between these two vectors is defined as:*

$$\|(u_1, u_2, \dots, u_n) - (v_1, v_2, \dots, v_n)\| = \|(u_1 - v_1, u_2 - v_2, \dots, u_n - v_n)\| =$$

$$\max_{k\in\{1,2,\dots,n\}}\{\|u_k - v_k\|\}.$$

Now that we have defined a notion of distance in the domain of multivariable functionals, we can now formally define what it means for a function vector to be a local extremum of a multivariable functional. It's defined as one would expect it to be.

Definition 2.2.4: *Let $J : E \subseteq \prod_{k=1}^{n} C^n[a,b] \to \mathbb{R}$ be a functional (here E is the domain of J). Then J has a **local minimum** at the function vector $(v_1, v_2, \dots, v_n)$ if:*

$$\forall\varepsilon > 0 \;\; \exists\delta > 0 \;\; \forall(u_1, u_2, \dots, u_n) \in E : \|(u_1, u_2, \dots, u_n) - (v_1, v_2, \dots, v_n)\| \le \delta,$$

$$J[u_1, u_2, \dots, u_n] \ge J[v_1, v_2, \dots, v_n].$$

*And J has a **local maximum** at $(v_1, v_2, \dots, v_n)$ if:*

$$\forall\varepsilon > 0 \;\; \exists\delta > 0 \;\; \forall(u_1, u_2, \dots, u_n) \in E : \|(u_1, u_2, \dots, u_n) - (v_1, v_2, \dots, v_n)\| \le \delta,$$

$$J[u_1, u_2, \dots, u_n] \le J[v_1, v_2, \dots, v_n].$$

Notice that this definition looks just like its one variable cousin Definition 1.2.6. Global extremums of multivariable functionals are defined just like you would expect them to be.

Definition 2.2.4: *Let $J : E \subseteq \prod_{k=1}^{n} C^n[a,b] \to \mathbb{R}$ be a functional. Then J has a* ***global minimum*** *at the function vector $(v_1, v_2, \dots, v_n)$ if:*

$$\forall (u_1, u_2, \dots, u_n) \in E, \qquad J[u_1, u_2, \dots, u_n] \geq J[v_1, v_2, \dots, v_n].$$

And J has a ***global maximum*** *at $(v_1, v_2, \dots, v_n)$ if:*

$$\forall (u_1, u_2, \dots, u_n) \in E, \qquad J[u_1, u_2, \dots, u_n] \leq J[v_1, v_2, \dots, v_n].$$

Now are now finally in a position to state and prove a theorem that gives a necessary condition for a multivariable functional to have an extremum at some function vector.

Theorem 2.2.5 (The Multivariable Euler-Lagrange Differential Equations): *Let J be a functional defined by:*

$$J[u_1(x), u_2(x), \dots, u_n(x)] = \int_a^b F\big(x, u_1(x), u_2(x), \dots, u_n(x), u_1'(x), u_2'(x), \dots, u_n'(x)\big)dx$$

where $F \in C^2[\mathbb{R}^{2n+1}]$. Let J's domain be the set of function vectors $(u_1, u_2, \dots, u_n) \in \prod_{k=1}^{n} C^n[a,b]$ that satisfy the boundary conditions:

$$\big(u_1(a), u_2(a), \dots, u_n(a)\big) = (A_1, A_2, \dots, A_n) \quad \textit{and}$$

$$\big(u_1(b), u_2(b), \dots, u_n(b)\big) = (B_1, B_2, \dots, B_n)$$

where $(A_1, A_2, \dots, A_n)$ and $(B_1, B_2, \dots, B_n)$ are two fixed vectors in $\mathbb{R}^n$. Then a necessary condition for J to have an extremum at $(v_1, v_2, \dots, v_n)$ is that the set of functions $v_1(x), v_2(x), \dots, v_n(x)$ satisfies the system of Euler-Lagrange partial differential equations:

$$\frac{\partial F}{\partial u_1}\big(x, u_1(x), u_2(x), \dots, u_n(x), u_1'(x), u_2'(x), \dots, u_n'(x)\big) - \frac{d}{dx}\left(\frac{\partial F}{\partial u_1'}\big(x, u_1(x), u_2(x), \dots, u_n(x), u_1'(x), u_2'(x), \dots, u_n'(x)\big)\right) = 0$$

$$\frac{\partial F}{\partial u_2}\big(x, u_1(x), u_2(x), \dots, u_n(x), u_1'(x), u_2'(x), \dots, u_n'(x)\big) - \frac{d}{dx}\left(\frac{\partial F}{\partial u_2'}\big(x, u_1(x), u_2(x), \dots, u_n(x), u_1'(x), u_2'(x), \dots, u_n'(x)\big)\right) = 0$$

$$\vdots$$

$$\frac{\partial F}{\partial u_n}\big(x, u_1(x), u_2(x), \dots, u_n(x), u_1'(x), u_2'(x), \dots, u_n'(x)\big) - \frac{d}{dx}\left(\frac{\partial F}{\partial u_n'}\big(x, u_1(x), u_2(x), \dots, u_n(x), u_1'(x), u_2'(x), \dots, u_n'(x)\big)\right) = 0$$

Remember, here $u_1, u_2, \dots, u_n$ are the variables of the functional J. So when we say that $v_1, v_2, \dots, v_n$ satisfy the above system of Euler-Lagrange partial differential equations we mean that if we plug in $v_1, v_2, \dots, v_n$ into the $u_1, u_2, \dots, u_n$ respectively, we will get that the above equations hold.

Proof: The proof of this theorem is very short and exactly analogous to how in calculus we proved that a necessary condition for a differentiable function of the form $F : \mathbb{R}^n \to \mathbb{R}$ to have an extremum at some point is that all of its partials are zero at that point. The idea is to apply the one variable Euler-Lagrange theorem (Theorem 1.3.3) that we proved in the previous chapter to each variable component of J.

Take an extremum $(v_1, v_2, \dots, v_n)$ of $J[u_1, u_2, \dots, u_n]$ (remember, here $u_1, u_2, \dots, u_n$ are the variables of our functional and $(v_1, v_2, \dots, v_n)$ is our point where the extremum lies). We want to prove that $v_1, v_2, \dots, v_n$ satisfy the above system of Euler-Lagrange partial differential equations. Fix $u_2 = v_2$, $u_3 = v_3, \dots, u_n = v_n$ but keep u_1 variable. Consider the one variable functional:

$$J[u_1, v_2, \dots, v_n] = \int_a^b F(x, u_1, v_2, \dots, v_n, u_1', v_2', \dots, v_n')dx.$$

Notice here that only u_1 is variable and the other $u_2, \dots, u_n$ are fixed to be $v_2, \dots, v_n$. So, this is a functional of the one variable $u_1(x)$. It is easy to see that since J has an extremum at $(v_1, v_2, \dots, v_n)$, the one variable functional $J[u_1, v_2, \dots, v_n]$ of the variable u_1 must have an extremum at $u_1 = v_1$ (this merely comes from the definition of extremums and norms defined above). So we can apply the one variable Euler-Lagrange theorem (Theorem 1.3.3) to the functional $J[u_1, v_2, \dots, v_n]$ to get that we must have that:

$$\frac{\partial F}{\partial u_1}\big(x, v_1(x), v_2(x), \dots, v_n(x), v_1'(x), v_2'(x), \dots, v_n'(x)\big)$$
$$- \frac{d}{dx}\left(\frac{\partial F}{\partial u_1'}\big(x, v_1(x), v_2(x), \dots, v_n(x), v_1'(x), v_2'(x), \dots, v_n'(x)\big)\right) = 0.$$

So $v_1, v_2, \dots, v_n$ satisfy the first Euler-Lagrange partial differential equation above. Similarly, if we fix $u_1 = v_2$, $u_3 = v_3, \dots, u_n = v_n$ but let u_2 be variable, repeating the above process gives us that $v_1, v_2, \dots, v_n$ must satisfy the second Euler-Lagrange partial differential equation above:

$$\frac{\partial F}{\partial u_2}\big(x, v_1(x), v_2(x), \dots, v_n(x), v_1'(x), v_2'(x), \dots, v_n'(x)\big)$$
$$- \frac{d}{dx}\left(\frac{\partial F}{\partial u_2'}\big(x, v_1(x), v_2(x), \dots, v_n(x), v_1'(x), v_2'(x), \dots, v_n'(x)\big)\right) = 0.$$

And doing this for each u_k gives that the $v_1, v_2, \dots, v_n$ must satisfy:

$$\frac{\partial F}{\partial u_k}\big(x, v_1(x), v_2(x), \dots, v_n(x), v_1'(x), v_2'(x), \dots, v_n'(x)\big)$$

$$-\frac{d}{dx}\left(\frac{\partial F}{\partial u_k'}\big(x, v_1(x), v_2(x), \ldots, v_n(x), v_1'(x), v_2'(x), \ldots, v_n'(x)\big)\right) = 0$$

for every $k \in \{1,2,\ldots,n\}$. So the $v_1, v_2, \ldots, v_n$ satisfy this system of Euler-Lagrange partial differential equations. With this we have proved the theorem.

■

Again, the proof of the above theorem isn't difficult and analogous to way the similar theorem is proved in multivariable calculus. The whole idea behind the necessary condition for a multivariable functional to have an extremum somewhere is that at that function vector" it satisfies the necessary condition for an extremum in each of its variables. This theorem will be crucial in the differential geometry chapters.

In analogy to how we defined the variational derivative (Definition 1.3.6) of a one variable functional in the previous chapter, we can define the variational gradient of a functional of several variables.

Definition 2.2.7: *Let J be a functional defined by:*

$$J[u_1, u_2, \ldots, u_n] = \int_a^b F(x, u_1, u_2, \ldots, u_n, u_1', u_2', \ldots, u_n')dx$$

where $F \in C^2[\mathbb{R}^{2n+1}]$. Let J's domain be the set of function vectors $(u_1, u_2, \ldots, u_n) \in \prod_{k=1}^{n} C^n[a,b]$ that satisfy the boundary conditions:

$$\big(u_1(a), u_2(a), \ldots, u_n(a)\big) = (A_1, A_2, \ldots, A_n) \quad and$$

$$\big(u_1(b), u_2(b), \ldots, u_n(b)\big) = (B_1, B_2, \ldots, B_n)$$

*where $(A_1, A_2, \ldots, A_n)$ and $(B_1, B_2, \ldots, B_n)$ are two fixed vectors in $\mathbb{R}^n$. Then the **variational gradient**, or **functional gradient**, of J is defined as:*

$$\nabla_\delta J[u_1, u_2, \ldots, u_n] = \left(\frac{\delta J}{\delta u_1}[u_1, u_2, \ldots, u_n], \frac{\delta J}{\delta u_2}[u_1, u_2, \ldots, u_n], \ldots, \frac{\delta J}{\delta u_n}[u_1, u_2, \ldots, u_n]\right) =$$

$$\left(\frac{\partial F}{\partial u_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_1}\right), \frac{\partial F}{\partial u_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_2}\right), \ldots, \frac{\partial F}{\partial u_n} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_n}\right)\right)$$

where each of the partials of F here are being evaluated at $(x, u_1, u_2, \ldots, u_n, u_1', u_2', \ldots, u_n')$. Basically what is done here is we take J, takes its one variable variational derivative in each of its variables u_k, and then form it into one big vector. Then this resulting vector is called the variational gradient.

As we will later see, the role that variational gradients play for functionals has a striking resemblance to the role that gradients play for multivariable functions. Already we can see this with the previous theorem by noticing that the necessary condition stated in Theorem 2.2.5 for the functional J in that theorem to have an extremum at $(v_1, v_2, \dots, v_n)$ can be rewritten as:

$$\nabla_\delta J[v_1, v_2, \dots, v_n] = 0.$$

On the right-hand side 0 is the zero vector in $\mathbb{R}^n$. This formula has a striking resemblance to the classical necessary condition for a differentiable multivariable function to have an extremum at some point.

Example 2.2.6: (The parametrized planar arclength minimizing curve problem) Take the multivariable functional:[12]

$$J[u(t), v(t)] = \int_a^b \sqrt{\left(u'(t)\right)^2 + \left(v'(t)\right)^2} dt$$

where its domain is the set of $(u, v) \in \prod_{k=1}^{2} C^n[a, b]$, such that ${u'}^2 + {v'}^2$ never vanishes and u and v satisfy the boundary conditions:

$$\left(u(a), v(a)\right) = (A_1, A_2),$$

$$\left(u(b), v(b)\right) = (B_1, B_2).$$

Another way to describe this domain is as the set of all continuously differentiable non-singular parametrized curves in $\mathbb{R}^2$ that satisfy the above boundary conditions (see the conventions section to see what "non-singular" means). We require that our curves are non-singular (the condition ${u'}^2 + {v'}^2 \neq 0$) to prevent them from having any kind of "pointy edges." Notice that what this functional does is it takes any such parametrized curve and returns its arc-length. Thus, this is another form of the arc-length functional that we've seen before (see Example 1.1.3). We could call this the "parametrized curve arc-length functional." So, omitting technical details we can colloquially say that finding the minimum of the above functional is the same thing as finding the shortest path between the two points (A_1, A_2) and (B_1, B_2).

As you probably know, the shortest path between any two points in the plane is the straight line that connects the two points (we will rigorously prove this fact in the differential geometry chapters. We will in fact prove this assertion in any dimensions n). Thus, the straight line is the minimum of the arc-length functional (it is a global minimum of J in fact). So, by the previous theorem we should get that the straight line between (A_1, A_2) and (B_1, B_2) satisfies the system of Euler-Lagrange differentiable equations:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0$$

[12] Here since the functional has only two variables, I decided to call them u and v instead of u_1 and u_2. The latter style of denoting variables by "u_k" with the index k is convenient (just like in multivariable calculus) when you have many variables involved. I also am using t rather than x as the variables for u and v.

$$\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) = 0$$

Let's check this! The straight line between (A_1, A_2) and (B_1, B_2) is parametrized by:

$$u(t) = A_1\left(1 - \frac{t-a}{b-a}\right) + B_1\frac{t-a}{b-a}$$

$$v(t) = A_2\left(1 - \frac{t-a}{b-a}\right) + B_2\frac{t-a}{b-a}$$

And plugging this into the system of Euler-Lagrange differentiable equations we see that the straight line does indeed satisfy this system of differential equations:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = \frac{\partial}{\partial u}\left(\sqrt{u'^2 + v'^2}\right) - \frac{d}{dx}\left(\frac{\partial}{\partial u'}\left(\sqrt{u'^2 + v'^2}\right)\right) = 0 - \frac{d}{dx}\left(\frac{u'}{\sqrt{u'^2 + v'^2}}\right) =$$

$$\frac{u''}{\sqrt{u'^2 + v'^2}} - \frac{u'^2 u''}{\sqrt{u'^2 + v'^2}^3} = \frac{0}{\sqrt{\left(\frac{B_1 - A_1}{b-a}\right)^2 + \left(\frac{B_2 - A_2}{b-a}\right)^2}} - \frac{\left(\frac{B_1 - A_1}{b-a}\right)^2 \cdot 0}{\sqrt{\left(\frac{B_1 - A_1}{b-a}\right)^2 + \left(\frac{B_2 - A_2}{b-a}\right)^2}^2} = 0$$

$$\frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) = \frac{\partial}{\partial v}\left(\sqrt{u'^2 + v'^2}\right) - \frac{d}{dx}\left(\frac{\partial}{\partial v'}\left(\sqrt{u'^2 + v'^2}\right)\right) = 0 - \frac{d}{dx}\left(\frac{u'}{\sqrt{u'^2 + v'^2}}\right) =$$

$$\frac{v''}{\sqrt{u'^2 + v'^2}} - \frac{v'^2 v''}{\sqrt{u'^2 + v'^2}^3} = \frac{0}{\sqrt{\left(\frac{B_1 - A_1}{b-a}\right)^2 + \left(\frac{B_2 - A_2}{b-a}\right)^2}} - \frac{\left(\frac{B_2 - A_2}{b-a}\right)^2 \cdot 0}{\sqrt{\left(\frac{B_1 - A_1}{b-a}\right)^2 + \left(\frac{B_2 - A_2}{b-a}\right)^2}^2} = 0$$

Our theory holds! Just for fun, let's compute this functional's variational gradient. In the above calculation we computed the variational derivative of J in each of its components:

$$\frac{\delta J}{\delta u} = \frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = \frac{u''}{\sqrt{u'^2 + v'^2}} - \frac{u'^2 u''}{\sqrt{u'^2 + v'^2}^3}$$

$$\frac{\delta J}{\delta v} = \frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) = \frac{v''}{\sqrt{u'^2 + v'^2}} - \frac{v'^2 v''}{\sqrt{u'^2 + v'^2}^3}$$

And so the variational gradient of J is given by:

$$\nabla_\delta J = \left(\frac{\delta J}{\delta u}, \frac{\delta J}{\delta v}\right) = \left(\frac{u''}{\sqrt{u'^2 + v'^2}} - \frac{u'^2 u''}{\sqrt{u'^2 + v'^2}^3}, \frac{v''}{\sqrt{u'^2 + v'^2}} - \frac{v'^2 v''}{\sqrt{u'^2 + v'^2}^3}\right)$$

Notice that everything here looks like multivariable calculus. This is a testament to the variational derivative and gradient notation!

As a last remark in this section, I would like to propose a new notation. Since functions of many variables play a central role in mathematics, it would be convenient to be able to write many variables in shorthand notation. If we are for example working with a function of n variables, we often write it as $F(x_1, x_2, \dots, x_n)$. Writing $x_1, x_2, \dots, x_n$ is rather cumbersome (especially if it turns out that the x_k's are not in fact variables but rather further functions of several variables). So, I propose the following **indexed argument notation**. Instead of writing $x_1, x_2, \dots, x_n$, the indexed argument notation writes $\{x_k\}_{k=1}^n$. So instead of writing:

$$F(x_1, x_2, \dots, x_n),$$

the indexed argument notation writes:

$$F(\{x_k\}_{k=1}^n).$$

As you can definitely see, this is much shorter than the normal notation.[13] The size difference however between normal and indexed argument notation is much more dramatic when you deal with compositions of multivariable functions. For example, suppose that now we want to plug in a set of multivariable functions $h_1(y_1, y_2, \dots, y_m)$, $h_2(y_1, y_2, \dots, y_m)$,…, $h_n(y_1, y_2, \dots, y_m)$ into the x_k's. Normally we would write this as:

$$F\big(h_1(y_1, y_2, \dots, y_m), h_2(y_1, y_2, \dots, y_m), \dots, h_n(y_1, y_2, \dots, y_m)\big).$$

But what the indexed argument notation does instead is: instead of writing out $h_1(y_1, y_2, \dots, y_m)$, $h_2(y_1, y_2, \dots, y_m)$,…, $h_n(y_1, y_2, \dots, y_m)$, it writes $\left\{h_k\left(\{y_j\}_{j=1}^m\right)\right\}_{k=1}^n$. Notice here that the indexed argument notation uses $\{\quad\}_{k=1}^n$ to enumerate over the h_k's and further $\{\quad\}_{k=1}^m$ to enumerate over the y_k's – the variables of the h_k's). So instead of the above expression, the indexed argument notation writes:

$$F\left(\left\{h_k\left(\{y_j\}_{j=1}^m\right)\right\}_{k=1}^n\right).$$

Here the difference between the normal notation and the indexed argument notation is much more considerable than in the last example. And if one has multiple expressions of this form on one line (as we will), the indexed argument notation can help greatly reduce the size of the formulas.

I will not heavily push this indexed argument notation since it's not standard. There are standard alternatives such as using vectors to denote a set of variables and functions. I will try to do everything in this book using normal notation. However, there will be moments in this book where the notation gets sooooo long that it will become nearly impossible to do the calculations without the use of some contracted notation. In those cases, I will use the indexed argument notation.

Let me start off by showing how the indexed argument notation can be used to write down both the functional and the system of Euler-Lagrange differential equations in Theorem 2.2.5 more

[13] Note, $\{x_k\}_{k=1}^n$ does not denote a finite sequence, it just uses the same curly bracket notation. To differentiate between indexed argument notation and finite sequences one has to look at the context.

concisely. In indexed argument notation, the functional in that theorem would be written down as:

$$J[\{u_k(x)\}_{k=1}^n] = \int_a^b F(x, \{u_k(x)\}_{k=1}^n, \{u_k'(x)\}_{k=1}^n)dx.$$

Notice that on the right-hand side, the indexed argument notation used separate $\{\quad\}_{k=1}^n$'s to enumerate over the u_k's and u_k''s. And the system of Euler-Lagrange partial differential equations in that theorem would be written down as:

$$\frac{\partial F}{\partial u_1}(x, \{u_k(x)\}_{k=1}^n, \{u_k'(x)\}_{k=1}^n) - \frac{d}{dx}\left(\frac{\partial F}{\partial u_1'}(x, \{u_k(x)\}_{k=1}^n, \{u_k'(x)\}_{k=1}^n)\right) = 0$$

$$\frac{\partial F}{\partial u_2}(x, \{u_k(x)\}_{k=1}^n, \{u_k'(x)\}_{k=1}^n) - \frac{d}{dx}\left(\frac{\partial F}{\partial u_2'}(x, \{u_k(x)\}_{k=1}^n, \{u_k'(x)\}_{k=1}^n)\right) = 0$$

$$\vdots$$

$$\frac{\partial F}{\partial u_n}(x, \{u_k(x)\}_{k=1}^n, \{u_k'(x)\}_{k=1}^n) - \frac{d}{dx}\left(\frac{\partial F}{\partial u_n'}(x, \{u_k(x)\}_{k=1}^n, \{u_k'(x)\}_{k=1}^n)\right) = 0$$

Section 3: Functionals over Spaces of Multivariable Functions Part 1

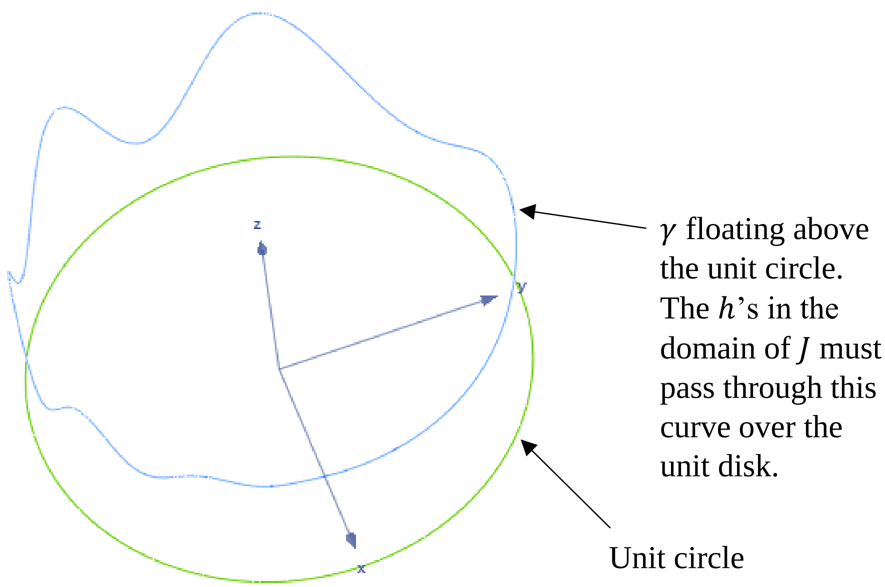
Sometimes in life, while strolling through the meadows of analysis, one encounters functionals whose domain is not a set of functions of one-variable as we've been seeing, but a set of multivariable functions. Indeed, in the previous chapter we've been studying functionals of the form $J[y(x)]$ where $y(x)$ is a function of one variable (the variable being x). But of course, why should we restrict our functionals' domains to be sets of one-variable functions when we could for example consider functionals of the form $J[h(x, y)]$ where $h(x, y)$ is a function of two variables. So let us develop a theory of functionals over spaces of multivariable functions.

Before we continuous, let me introduce a standard notation that we will often use in this book. If $\mathcal{F}(x, y)$ is a function (real or vector valued) of x and y, let $\mathcal{F}_x$ and $\mathcal{F}_y$ denote the partials $\frac{\partial \mathcal{F}}{\partial x}$ and $\frac{\partial \mathcal{F}}{\partial y}$ respectively. This is the "Newtonian notation" for differentiation and it often makes equations look shorter compared to the Leibniz partial notation. As this notation would suggest, $\mathcal{F}_{xx}$, $\mathcal{F}_{xy}$, and $\mathcal{F}_{yy}$ denote the partials $\frac{\partial^2 \mathcal{F}}{\partial x^2}$, $\frac{\partial^2 \mathcal{F}}{\partial x \partial y}$, and $\frac{\partial^2 \mathcal{F}}{\partial y^2}$ respectively. Similar sort of thing goes for third order partials, fourth order partial, etc. This notation will often help us make our equations shorter in the cases when they would get rather long and bulky if we used the Leibniz notation instead.

Functionals over spaces of multivariable functions have significant application in mathematics and physics. They for example will play a central role when we study the differential geometry of surfaces. As an example of a functional whose domain is a set of multivariable functions, consider the following functional:

$$J[h(x,y)] = \iint_{\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1\}} \sqrt{1 + \left(h_x(x,y)\right)^2 + \left(h_y(x,y)\right)^2}\, dxdy$$

where h_x and h_y denote the partials $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ respectively. Let us say that the domain of this functional J is the set of all functions $h(x,y)$ that are continuously differentiable over the closed unit disk $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ (we require that the h's are *continuously* differentiable functions so as to guarantee that the above integral makes sense) and that they satisfy the boundary conditions of passing through some non-singular space C^∞ curve $\gamma(t)$ in $\mathbb{R}^3$ over the unit disk:



Notice that this integral is the "surface area integral." So, basically what this functional does is it takes a continuously differentiable function $h(x,y)$ that satisfies the above boundary condition and outputs the surface area of its graph over the closed unit disk. Minimizing the above functional will be the subject of our study of minimal surfaces in the differential geometry chapters since minimal surfaces are defined as surfaces that minimize surface area.

The theory of functionals over spaces of multivariable functions is developed in almost exactly the same fashion as we developed it for functionals over spaces of functions of one variable in the last chapter. As a result, many of the details in the development of this more general theory are pretty much the same or closely resemble what we did in the previous chapter. For this reason some of the details in this section are briskly covered since we've pretty much already seen how their done in the lower dimensional case.

Since we are covering more general types of functionals, the domains of our functionals will tend to be subsets of $C^m[E]$ where $E \subseteq \mathbb{R}^n$ (see Notation 1.2.1). So we need to define a norm on these spaces. Let's define it as follows:

Definition 2.3.1: *Let E be a compact subset of $\mathbb{R}^n$ and let $h(x_1, x_2, \dots, x_n) \in C^m[E]$. Then the **norm** of h in this space is defined as:*

$$\|h\| = \sum_{k_1,k_2,\dots,k_n \in \mathbb{Z}_+ \cup \{0\}: k_1+k_2+\cdots+k_n \le m} \max_{(x_1,x_2,\dots,x_n)\in E} \left| \frac{\partial^{k_1+k_2+\cdots+k_n} h}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} (x_1, x_2, \dots, x_n) \right|$$

$$= \sum_{k_1,k_2,\dots,k_n \in \mathbb{Z}_+ \cup \{0\}: k_1+k_2+\cdots+k_n \le m} \max_{(x_1,x_2,\dots,x_n)\in E} \left| \frac{\partial^{\sum_{j=1}^n k_j} h}{\prod_{j=1}^n \partial x_j^{k_j}} (x_1, x_2, \dots, x_n) \right|.$$

In other words, we take all possible partials $\frac{\partial^{\sum_{j=1}^n k_j} h}{\prod_{j=1}^n \partial x_j^{k_j}}$ of h of order less than or equal to m and add up all of their maxes on E. The fact that these maxes exist is guaranteed by the Extreme Value Theorem and the fact that E is compact.

Similarly as we did before, we define the distance between two functions in $C^m[E]$ as the norm of their difference:

Definition 2.3.2: *Let E be a compact subset of $\mathbb{R}^n$ and let $h(x_1, x_2, \dots, x_n), g(x_1, x_2, \dots, x_n) \in C^m[E]$. Then the **distance** between h and g is defined as:*

$$\|h - g\| = \sum_{k_1,k_2,\dots,k_n \in \mathbb{Z}_+ \cup \{0\}: k_1+k_2+\cdots+k_n \le m} \max_{(x_1,x_2,\dots,x_n)\in E} \left| \frac{\partial^{\sum_{j=1}^n k_j} h}{\prod_{j=1}^n \partial x_j^{k_j}} - \frac{\partial^{\sum_{j=1}^n k_j} g}{\prod_{j=1}^n \partial x_j^{k_j}} \right|.$$

Notice that Definition 1.2.3 and Definition 1.2.4 that we made in the previous chapter are special cases of the above two definitions.

Local and global extremums are defined as one would expect them to be:

Definition 2.3.3: *Let $J[h(x_1, x_2, \dots, x_n)]$ be a functional whose domain is a subset of $C^n[E]$ where $E \subset \mathbb{R}^n$ is a compact subset. Then J has a **local minimum** at the function $h_1(x_1, x_2, \dots, x_n) \in dom(J)$ if:*

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall g \in dom(J) : \|g - h\| \le \delta, \qquad J[g] \ge J[h_1].$$

*J has a **local maximum** at the function $h_1(x_1, x_2, \dots, x_n) \in dom(J)$ if:*

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall g \in dom(J) : \|g - h\| \le \delta, \qquad J[g] \le J[h_1].$$

Definition 2.3.4: *Let $J[h(x_1, x_2, \dots, x_n)]$ be a functional whose domain is a subset of $C^n[E]$ where $E \subset \mathbb{R}^n$ is a compact subset. Then J has a **global minimum** at the function $h_1(x_1, x_2, \dots, x_n) \in dom(J)$ if:*

$$\forall g \in dom(J), \qquad J[g] \geq J[h_1].$$

*J has a **global maximum** at the function $h_1(x_1, x_2, \ldots, x_n) \in dom(J)$ if:*

$$\forall g \in dom(J), \qquad J[g] \leq J[h_1].$$

Notice that these definitions of local and global extrema of functionals are defined almost exactly the same way as we defined them in the previous chapter. As on could guess, we define smooth linear flows in $C^n[E \subseteq \mathbb{R}^n]$ similarly to the way we did in the previous chapter.

Definition 2.3.5: *Let $E \subseteq \mathbb{R}^n$. Then an **n-smooth linear flow** of multivariable functions is a function $\Lambda : E \times [t_0, t_1] \to \mathbb{R}$ of the form:*

$$\Lambda(x_1, x_2, \ldots, x_n, t) = h_1(x_1, x_2, \ldots, x_n) + h_2(x_1, x_2, \ldots, x_n) \cdot t$$

where h_1 and h_2 are functions in $C^n[a, b]$.

Again, for any n-smooth linear flow of multivariable functions $\Lambda(x_1, x_2, \ldots, x_n, t)$, we will not think of Λ as merely a function of $n + 1$ variables. As with the case of linear flows of curves in the previous chapter, we will look at $\Lambda(x_1, x_2, \ldots, x_n, t)$ as a function of the n variables $x_1, x_2, \ldots, x_n$ for every fixed t. Thus, our perspective will be that $\Lambda(x_1, x_2, \ldots, x_n, t)$ represents a sort of movement of functions through the space $C^n[E]$ as t runs across the interval $[t_0, t_1]$ from time t_0 to time t_1. Or more artistically speaking, $\Lambda(x_1, x_2, \ldots, x_n, t)$ is a "flow" of n-variable functions. The next example helps illustrate this point of view.

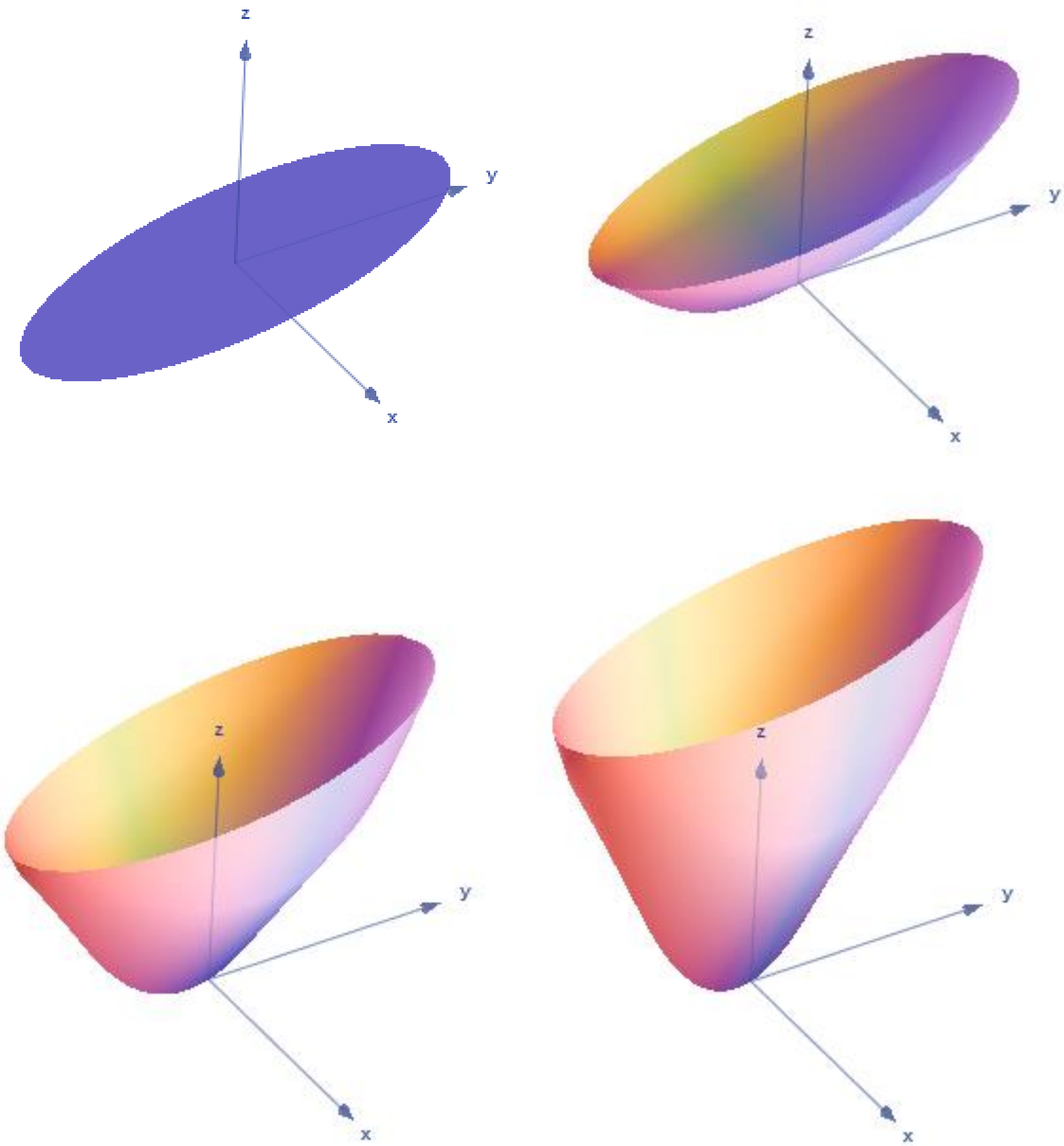
Example 2.3.6: Take the 3-smooth linear flow of multivariable functions,

$$\Lambda : \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \times [0, 1] \to \mathbb{R}$$

defined by:

$$\Lambda(x, t) = \left(\frac{1}{2}x + \frac{1}{3}y\right) + (2x^2 + 2y^2)t$$

(in this example our E is the closed unit circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $[t_0, t_1] = [0, 1]$). Basically what this linear flow does is it flows from the plane $\frac{1}{2}x + \frac{1}{3}y$ to the cup like surface $\left(\frac{1}{2}x + \frac{1}{3}y\right) + (2x^2 + 2y^2)$ over the closed unit circle. The following images show how the flow Λ looks like at times $t = 0, 0.33, 0.66, 1$ respectively:



Thus Λ flows (or "moves") from the function $\frac{1}{2}x + \frac{1}{3}y$ to the function $\left(\frac{1}{2}x + \frac{1}{3}y\right) + (2x^2 + 2y^2)$ as the time variable t runs from $t = 0$ to $t = 1$.

The canonical form of an n-smooth linear flow of multivariable functions is defined similarly to its cousin in the previous chapter.

Definition 2.3.7: *Let $\Lambda : E \times [t_0, t_1] \to \mathbb{R}$ be an n-smooth linear flow of multivariable functions as defined in Definition 2.3.5. Then the* ***canonical form*** *of the flow $\Lambda(x_1, x_2, \dots, x_n, t)$ is the function $\widetilde{\Lambda} : [t_0, t_1] \to C^n[E]$ defined by:*

$$\widetilde{\Lambda}(t) = \Lambda(x_1, x_2, \dots, x_n, t)$$

where in this case $\Lambda(x_1, x_2, \dots, x_n, t)$ *on the right-hand side denotes the multivariable function of the variables* $x_1, x_2, \dots, x_n$ *for every fixed* t.

Just like for linear flows of curves in the previous chapter, canonical forms of n-smooth linear flows of multivariable functions give more of a topological approach to the study of these kinds of flows. Analogously, these flows also satisfy the property that they flow through $C^n[E]$ continuously. This is stated precisely in the following theorem.

Theorem 2.3.8: *Let* $\Lambda : E \times [t_0, t_1] \to \mathbb{R}$, *where* $E \subseteq R^n$ *is a compact set, be an* n*-smooth linear flow of multivariable functions as defined in Definition 2.3.5. Then* $\widetilde{\Lambda}(t)$ *is a continuous function.*

I will leave the proof of this theorem to the reader since its proof is almost identical in nature to its analog Theorem 1.2.12. Just mimic the same steps as in the proof of that theorem and the proof of this theorem should come out pretty quickly.

Section 4: Functionals over Spaces of Multivariable Functions Part 2

In this section we will prove yet another exciting generalization of the Basic Euler-Lagrange Theorem from the previous chapter. This generalization deals with the necessary condition for a functional of multivariable functions to have an extremum at some function.

In the previous chapter we often called functions of one variable, such as $y(x)$, "curves" since their graphs resembled actual curves. In the study of functionals of multivariable functions we can do a similar thing by colloquially referring to functions of several variables as "surfaces" since their graphs look like surfaces or hypersurfaces (surfaces with dimension bigger than 2). The result is that we can informally call a functional of the form $J[h(x_1, x_2, \dots, x_n)]$ a functional of surfaces. Thus we can say that the main difference between what we are going to do in this section and what we did in the last chapter is that we will be varying surfaces rather than curves in order to find the extremums of our functionals.

In what follows, I will present results in the following pattern: the version for functionals of the form $J[h(x,y)]$ and then the version for functionals of the more general form $J[h(x_1, x_2, \dots, x_n)]$. The reason for doing this is that the calculations involving $J[h(x,y)]$ are easier to visualize than the calculations for the more general case $J[h(x_1, x_2, \dots, x_n)]$. So by doing the $J[h(x,y)]$ case first, when we do get to the calculations for $J[h(x_1, x_2, \dots, x_n)]$ we will already have a bit of visual experience backing us up while we try to make sense out of all the symbols involved in the calculations dealing with the more general case $J[h(x_1, x_2, \dots, x_n)]$.

In this section we will be studying necessary conditions for extremums of functionals of the form:

$Equation$ 2.4.1:

$$J[h(x,y)] = \iint_{\Omega} F(x, y, h, h_x, h_y)\,dx\,dy$$

where again h_x and h_y denote the partials $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ respectively. An example of such a functional is the surface area functional given at the beginning of the previous section. With the surface area functional, we had that $F(x, y, h, h_x, h_y) = \sqrt{1 + h_x^2 + h_y^2}$.

As in the previous chapter, in order to derive the Euler-Lagrange differential equation for functionals of the above form, we will need the following simple but powerful lemma:

The proof of the following two lemmas come from Gelfand and Fomin's book on the calculus of variations.

Lemma 2.4.2 (Two-variables Version): *Let m be a fixed non-negative integer and suppose that $\Omega \subseteq \mathbb{R}^2$ is a non-empty open subset of $\mathbb{R}^2$. Suppose also that $\alpha(x, y) \in C^0[\Omega]$ is a continuous function such that for $\forall h \in C^m[\Omega]$ that satisfies the boundary condition:*

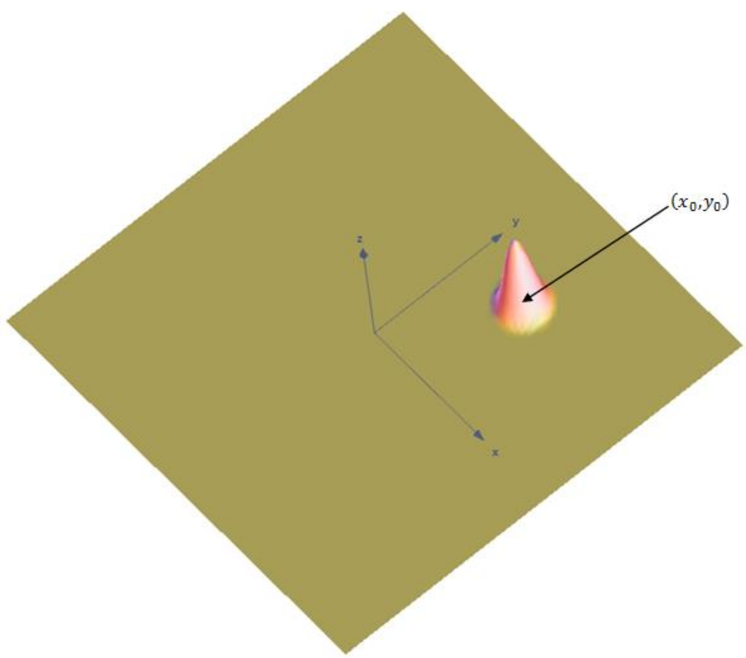
$$h(x, y) = 0 \quad for\ any \quad (x, y) \in \partial\Omega,$$

(in other words, h vanishes on the boundary of Ω) the following integral is equal to zero:

$$\iint_{\Omega} \alpha(x, y) h(x, y) dx dy = 0.$$

Then $\alpha(x) = 0$ on all of $x \in \Omega$.

Proof: This lemma is proved in almost exactly the same way as we proved its one-dimensional cousin Lemma 1.3.1 in the previous chapter. The whole idea behind this lemma is that if α was nonzero at some point $(x_0, y_0) \in \Omega$, then by α's continuity we would know that locally to (x_0, y_0) α would also be nonzero. From there we would be able to construct a function $h \in C^m[\Omega]$ that is zero everywhere except near x_0 where h spikes up:



Then intuitively in terms of volume we could see that with this h the integral $\iint_\Omega \alpha(x,y)h(x,y)dxdy$ would be nonzero, which would be a contradiction since we said that for all h that satisfy the above boundary condition the integral $\iint_\Omega \alpha(x,y)h(x,y)dxdy = 0$.

I will quickly cover the details in the proof of this lemma since the details are pretty much analogous to what we did in Lemma 1.3.1. This lemma is proved by contradiction. Suppose that there was a point (x_0, y_0) such that $\alpha(x_0, y_0) \neq 0$. Without loss of generality, suppose that $\alpha(x_0, y_0) > 0$ (the case $\alpha(x_0, y_0) < 0$ is dealt with similarly). Notice that in this lemma we don't even need to bother with the boundary of our set since we agreed to let Ω be open. Then, by the continuity of α we know that α will be positive within some ball of radius $r > 0$ centered at (x_0, y_0).

Now, an equation (one of many possibilities) for an m-times continuously differentiable positive spike around (x_0, y_0) that vanishes outside the ball of radius r centered at (x_0, y_0) is given by:

$$h(x,y) = \begin{cases} 0 \quad if \quad (x,y) \notin B_r(x_0,y_0) \\ \left(\frac{1}{2}\cos\left(\frac{\pi}{r^2}((x-x_0)^2+(y-y_0)^2)\right)+\frac{1}{2}\right)^{m+1} \quad if \quad x \in B_r(x_0,y_0) \end{cases}$$

(I highly recommend graphing this h for different values of r, x_0, and y_0 to see what it looks like). Let's see why with this h we get that:

$$\iint_\Omega \alpha(x,y)h(x,y)dxdy = \iint_{B_r(x_0,y_0)} \alpha(x,y)h(x,y)dxdy > 0$$

This integral is positive (in particularly non-zero) since $\alpha(x,y)h(x,y) > 0$ in $B_r(x_0, y_0)$, $B_r(x_0, y_0)$ has positive area, and $\alpha(x,y)h(x,y)$ is zero everywhere else. But this inequality then gives us the contradiction that we need since by the hypothesis of the lemma this integral is supposed to be equal to zero. So, α must be zero on all of Ω.

■

A version with the above lemma where there are n variables involved goes as follows:

Lemma 2.4.3 (n-Variables Version): *Let m be a fixed non-negative integer and suppose that $\Omega \subseteq \mathbb{R}^n$ is a non-empty open subset of $\mathbb{R}^n$. Suppose also that $\alpha(x,y) \in C^0[\Omega]$ is a continuous function such that for $\forall h \in C^m[\Omega]$ that satisfies the boundary conditions:*

$$h(x_1, x_2, \ldots x_n) = 0 \quad for\ any \quad (x_1, x_2, \ldots x_n) \in \partial\Omega,$$

(in other words, h vanishes on the boundary of Ω) the following integral is equal to zero (in this lemma, let $\int$ denote an integral over a region in $\mathbb{R}^n$):

$$\int_\Omega \alpha(x_1, x_2, \ldots x_n)h(x_1, x_2, \ldots x_n)\prod_{k=1}^{n} dx_k = 0.$$

Then $\alpha(x) = 0$ on all of $x \in \Omega$.

Proof: This is proved just like the previous lemma except that here you will need the following equation for an m-times continuously differentiable positive spike around a fixed point $(x_{0_1}, x_{0_2}, \dots, x_{0_n}) \in \Omega$ that vanishes outside the ball of radius $r > 0$ centered at $(x_{0_1}, x_{0_2}, \dots, x_{0_n})$:

$$h(x_1, x_2, \dots, x_n) = \begin{cases} 0 \quad if \quad (x_1, x_2, \dots, x_n) \notin B_r(x_{0_1}, x_{0_2}, \dots, x_{0_n}) \\ \left(\frac{1}{2}\cos\left(\frac{\pi}{r^2}\sum_{k=1}^{n}\left(x_k - x_{0_k}\right)^2\right) + \frac{1}{2}\right)^{m+1} \quad if \quad (x_1, x_2, \dots, x_n) \notin B_r(x_{0_1}, x_{0_2}, \dots, x_{0_n}) \end{cases}.$$

■

As before, the above two lemmas do have interesting interpretations in terms of inner products on a vector space of functions. For example, Lemma 2.4.2 can be interpreted as follows. Let $\Omega \subseteq \mathbb{R}^2$ be a non-empty open subset of $\mathbb{R}^2$ and let m be some fixed non-negative integer. Consider the real vector space:

$$V = \{h \in C^m[\Omega] : h(x, y) = 0 \;\; if \;\; (x, y) \in \partial\Omega\}.$$

In other words, this is the vector space of m-times continuously differentiable functions on Ω that vanish on the boundary of Ω. Notice that this is a subspace of the bigger vector space of all <u>continuous</u> functions on Ω:

$$X = C^0[\Omega]$$

coupled with the inner product:

$$\langle f, g\rangle = \iint\limits_{\Omega} f(x, y)g(x, y)dxdy.$$

Then what Lemma 2.4.2 says is that if $\alpha \in X$ is a vector such that $\langle \alpha, h\rangle = 0$ for any vector $h \in V$, then α is equal to the zero vector. In other words, the only vector in X that is orthogonal to the subspace V is the zero vector. Symbolically this is written as $V^\perp = \{0\}$. This is a strong result that says a lot about how the subspace V sits in X. However, we won't use this interpretation of Lemma 2.4.2 anywhere in this book.

We are finally in a position to derive the Euler-Lagrange partial differential equation for functionals of the form in Equation 2.4.1. In the proof of the following theorem, we will be using Green's Theorem on the region of integration Ω. Since it's really difficult to describe all types of regions that Green's Theorem applies to, I will not attempt to give a precise statement about what Ω looks like. This is why in the statement of the following theorem, primarily I'm just going to state that our region of integration Ω is a region that Green's Theorem is applicable to and go from there. Most "fat" compact regions that we encounter in life will fit the description of Ω in the following theorem.

In the following theorem we are going to need to precisely define a boundary condition for the functions $h(x, y)$ in the domain of our functional J. In the case of functionals over the space of curves $y(x)$, we had that our curves satisfied boundary condition of the form $y(a) = A$ and $y(b) = B$ where A and B were two real numbers. Here our boundary conditions for our "surfaces" $h(x, y)$ will be that the pass through some C^2 non-singular space curve $\gamma(t)$ in $\mathbb{R}^3$ above the edge of the set on which they are defined, which we will call Ω. The reason that we want $\gamma(t)$ to be non-singular is to prevent it from having any kind of weird sharp edges (curves can have such edges if you let their derivatives vanish). And the reason that we want it to be C^2 is because it isn't hard to see that if it wasn't C^2 then there just wouldn't be any C^2 h's in the domain of J that satisfy this boundary condition since the h's are defined over the boundary of Ω.

Theorem 2.4.4 (The Euler-Lagrange Partial Differential Equation for Functionals of Two-Variable Functions): *Let $\Omega \subseteq \mathbb{R}^2$ be a compact subset of $\mathbb{R}^2$ that Green's Theorem applies to. Suppose that Ω is the closure of its interior and that its boundary ∂Ω is a set of zero Jordan content.[14] Suppose also that we have a function of the form $f : \partial\Omega \to \mathbb{R}$ and that the set:*

$$\left\{\left(x, y, f(x, y)\right) \in \mathbb{R}^3 : (x, y) \in \partial\Omega\right\}$$

can be parametrized by a non-singular C^2 curve $\gamma(t)$ in $\mathbb{R}^3$. This will serve as our boundary conditions for the function in the domain of our functional. Now, let J be a functional of the form:

$$J[h(x, y)] = \iint_{\Omega} F\left(x, y, h, h_x, h_y\right) dx dy$$

where $F \in C^2[\mathbb{R}^5]$. Let J's domain be the set of "surfaces" $h(x, y) \in C^2[\Omega]$ that satisfy the boundary conditions:

$$h(x, y) = f(x, y) \quad if \quad (x, y) \in \partial\Omega$$

(in other words, the h's pass through the curve $\gamma(t)$ above ∂Ω). Then a necessary condition for the surface $H(x, y)$ to be a local extremum of the functional J is that it satisfies the Euler-Lagrange partial differential equation on Ω:

$$\frac{\partial F}{\partial h}\left(x, y, h, h_x, h_y\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\left(x, y, h, h_x, h_y\right)\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\left(x, y, h, h_x, h_y\right)\right) = 0.$$

Omitting the arguments of F, this equation takes the nicer to look at form:

$$\frac{\partial F}{\partial h} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right) = 0.$$

---

[14] ∂Ω being of a set of zero Jordan content means that the integral of any continuous function over ∂Ω is always zero. In other words, ∂Ω is a negligible set when it comes to integration. Requiring ∂Ω to be a set of zero Jordan content is a small technicality that we will need in order to rigorously apply Lemma 2.4.2 towards the end of the proof of this theorem.

Proof: Suppose that $H(x,y)$ is an extremum of the functional J. We want to prove that $H(x,y)$ satisfies the above Euler-Lagrange partial differential equation. Take any function $g \in C^2[\Omega]$ that satisfies the boundary condition of vanishing on $\partial\Omega$ and form the linear flow $\Lambda :$ $\Omega \times [-1,1] \to \mathbb{R}$ defined by:

$$\Lambda(x,y,t) = H(x,y) + g(x,y) \cdot t.$$

Notice that this Λ is a "linear flow of surfaces" that passes through our extremum $H(x,y)$ at time $t = 0$ (meaning $\Lambda(x,y,0) = H(x,y)$) and flows with constant speed $g(x,y)$ (meaning $\frac{\partial \Lambda}{\partial t} = g(x,y)$). Now consider the real valued function:

$$\mathcal{G}(t) = J[\Lambda(x,y,t)] = \iint_{\Omega} F\left(x,y,\Lambda(x,y,t),\frac{\partial \Lambda}{\partial x}(x,y,t),\frac{\partial \Lambda}{\partial y}(x,y,t)\right) dxdy.$$

Here all we did was for each time $t \in [-1,1]$ we plugged in the surface $\Lambda(x,y,t)$ into the functional J. Now just like in the proof of Theorem 1.3.3, let us notice that since $H(x,y)$ is an extremum of J, the function $\mathcal{G}(t)$ should have a local extremum at $t = 0$ since out flow Λ passes through H at time $t = 0$. This would then imply that $\frac{d}{dt}\left(\mathcal{G}(t)\right)\Big|_{t=0} = 0$. Let's state and prove this in a lemma.

Lemma 2.4.5: *Our function $\mathcal{G}(t)$ above is differentiable and its derivative at $t = 0$ is equal to zero:*

$$\mathcal{G}'(0) = \frac{d}{dt}\left(J[\Lambda(x,y,t)]\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\iint_{\Omega} F\left(x,y,\Lambda(x,y,t),\frac{\partial \Lambda}{\partial x}(x,y,t),\frac{\partial \Lambda}{\partial y}(x,y,t)\right) dxdy\right)\Bigg|_{t=0} = 0.$$

Proof: This lemma is proved just like how we proved Lemma 1.3.5 in the proof of the basic Euler-Lagrange differential equation except that here we are dealing with more variables. We have that the function $\mathcal{G}(t)$ is differentiable because we can carry the time derivative under the integral sign (since the above integrand is continuously differentiable and our domain of integration is compact). Now in order to prove that the above derivative is equal to zero, it is sufficient to just show that $t = 0$ is a local extremum of our function $\mathcal{G}(t)$. The fact that $\mathcal{G}'(0) = 0$ will then follow from the well-known fact that the derivative of a function at an extremum is equal to zero.

Let's suppose that $H(x,y)$ is a local minimum of the functional J. The proof of this lemma in the case of when $H(x,y)$ is a local maximum of the functional J is similar. Then by definition there exists a $\delta > 0$ such that for any $h \in \mathrm{dom}(J)$ such that $\|h - H\| \leq \delta$,

$$J[h(x,y)] \geq J[H(x,y)].$$

Now, by Theorem 2.3.8 we have that our 2-smooth linear flow $\Lambda(x, y, t)$ flows continuously through the space $\text{dom}(J) \subseteq C^2[\Omega]$. So there exists a $\Delta > 0$ such that for any time $t \in [-1, 1]$ such that $|t - 0| \leq \Delta$,

$$\|\Lambda(x, y, t) - \Lambda(x, y, 0)\| = \|\Lambda(x, y, t) - H(x, y)\| \leq \delta.$$

This means that for any $t \in [-\Delta, \Delta]$,

$$J[\Lambda(x, y, t)] \geq J[H(x, y)] = J[\Lambda(x, y, 0)].$$

Or in other words: for any $t \in [-\Delta, \Delta]$,

$$\mathcal{G}(t) \geq \mathcal{G}(0).$$

So $t = 0$ is a local extremum of $\mathcal{G}(t)$ and thus $\mathcal{G}'(0) = 0$.

■

Now we are ready to do the exciting calculation that proves the theorem. In the above lemma we proved that:

$$\frac{d}{dt}\left(\iint_{\Omega} F\left(x, y, \Lambda(x, y, t), \frac{\partial \Lambda}{\partial x}(x, y, t), \frac{\partial \Lambda}{\partial y}(x, y, t)\right) dxdy\right)\Bigg|_{t=0} = 0.$$

Carry the above $\frac{d}{dt}$ derivative under the integral sign to get that:

$$\frac{d}{dt}\left(\iint_{\Omega} F\left(x, y, \Lambda(x, y, t), \frac{\partial \Lambda}{\partial x}(x, y, t), \frac{\partial \Lambda}{\partial y}(x, y, t)\right) dxdy\right)\Bigg|_{t=0}$$

$$= \iint_{\Omega} \frac{d}{dt}\left(F\left(x, y, \Lambda(x, y, t), \frac{\partial \Lambda}{\partial x}(x, y, t), \frac{\partial \Lambda}{\partial y}(x, y, t)\right)\right)\Bigg|_{t=0} dxdy =$$

$$\iint_{\Omega} \left(\frac{\partial F}{\partial h}\left(x, y, \Lambda(x, y, 0), \frac{\partial \Lambda}{\partial x}(x, y, 0), \frac{\partial \Lambda}{\partial y}(x, y, 0)\right)\frac{\partial \Lambda(x, y, 0)}{\partial t}\right.$$
$$+ \frac{\partial F}{\partial h_x}\left(x, y, \Lambda(x, y, 0), \frac{\partial \Lambda}{\partial x}(x, y, 0), \frac{\partial \Lambda}{\partial y}(x, y, 0)\right)\frac{\partial \Lambda(x, y, 0)}{\partial x \partial t}$$
$$\left.+ \frac{\partial F}{\partial h_y}\left(x, y, \Lambda(x, y, 0), \frac{\partial \Lambda}{\partial x}(x, y, 0), \frac{\partial \Lambda}{\partial y}(x, y, 0)\right)\frac{\partial \Lambda(x, y, 0)}{\partial y \partial t}\right) dxdy = 0.$$

Wow, our equations are getting quite long (wait till we do this in n-variables!). Now let us plug in the equation that we had for our flow: $\Lambda(x, y, t) = H(x, y) + g(x, y) \cdot t$ into the above equation. We will get that:

$$\iint_\Omega \left(\frac{\partial F}{\partial h}\left(x, y, H(x,y), \frac{\partial H}{\partial x}(x,y), \frac{\partial \mathrm{H}}{\partial y}(x,y)\right) g(x,y) \right.$$
$$+ \frac{\partial F}{\partial h_x}\left(x, y, H(x,y), \frac{\partial H}{\partial x}(x,y), \frac{\partial \mathrm{H}}{\partial y}(x,y)\right) \frac{dg}{\partial x}(x,y)$$
$$\left. + \frac{\partial F}{\partial h_y}\left(x, y, H(x,y), \frac{\partial H}{\partial x}(x,y), \frac{\partial \mathrm{H}}{\partial y}(x,y)\right) \frac{dg}{\partial y}(x,y) \right) dxdy = 0.$$

Let us for the time being stop writing the arguments of the partials of F so as to make the equations shorter. Rewriting the above equation but omitting the arguments of the partials of F gives:

$$Equation\ 2.4.6\text{:}\quad \iint_\Omega \left(\frac{\partial F}{\partial h} \cdot g(x,y) + \frac{\partial F}{\partial h_x} \cdot \frac{dg}{\partial x}(x,y) + \frac{\partial F}{\partial h_y} \cdot \frac{\partial g}{\partial y}(x,y) \right) dxdy = 0.$$

Now that we have this nice equation, let's get rid of the annoying $\partial/\partial x$ and $\partial/\partial y$ partials of g. How did we do this analogous step in our proof of Theorem 1.3.3? We integrated by parts. Here we can't integrate by parts, but what we can do is apply Green's Theorem. Take the $\iint_\Omega \left(\frac{\partial F}{\partial h_x} \cdot \frac{dg}{\partial x} + \frac{\partial F}{\partial h_y} \cdot \frac{\partial g}{\partial y} \right) dxdy$ term in the above integral. Applying the identity (notice that this has a striking resemblance to integration by parts):

$$u \frac{\partial v}{\partial w} = \frac{\partial}{\partial w}(uv) - \frac{\partial u}{\partial w} v$$

to each term $\frac{\partial F}{\partial h_x} \cdot \frac{dg}{\partial x}$ and $\frac{\partial F}{\partial h_y} \cdot \frac{\partial g}{\partial y}$ gives us that we can rewrite our term as:

$$\iint_\Omega \left(\frac{\partial F}{\partial h_x} \cdot \frac{dg}{\partial x} + \frac{\partial F}{\partial h_y} \cdot \frac{\partial g}{\partial y} \right) dxdy$$
$$= \iint_\Omega \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x} \cdot g \right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y} \cdot g \right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x} \right) g - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y} \right) g \right) dxdy.$$

Applying Green's Theorem to the $\iint_\Omega \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x} \cdot g \right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y} \cdot g \right) \right) dxdy$ portion of the integral turns the above quantity into:

$$\oint_{\partial\Omega} \left(-\frac{\partial F}{\partial h_y} \cdot g, \frac{\partial F}{\partial h_x} \cdot g \right) \cdot d\vec{x} - \iint_\Omega \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x} \right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y} \right) \right) g(x,y) dxdy.$$

Since g vanishes on $\partial\Omega$, we get that the above contour integral is equal to zero and so we're only left with:

$$-\iint_{\Omega}\left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right)\right)g(x,y)dxdy.$$

So, by an application of Green's Theorem we have proved that our term is equal to:

$$\iint_{\Omega}\left(\frac{\partial F}{\partial h_x}\cdot\frac{dg}{\partial x}+\frac{\partial F}{\partial h_y}\cdot\frac{\partial g}{\partial y}\right)dxdy=-\iint_{\Omega}\left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right)\right)g(x,y)dxdy.$$

We basically "integrated by parts" the integral on the left via Green's Theorem. Plugging our term back into Equation 2.4.6 gives that:

$$\iint_{\Omega}\left(\frac{\partial F}{\partial h}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right)\right)g(x,y)dxdy=0.$$

Since the boundary of our region of integration Ω has zero Jordan content, the above integral doesn't get affected if we only integrate over Ω's interior:

$$\iint_{\Omega^{int}}\left(\frac{\partial F}{\partial h}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right)\right)g(x,y)dxdy=0.$$

We rewrote the above integral in this way because we are about to apply Lemma 2.4.2 to this integral, and technically for this lemma to be applicable we need our region of integration to be open. Now, let's write out the arguments of the partials of F so as to make the above equation a little bit more explicit:

$$\iint_{\Omega^{int}}\left(\frac{\partial F}{\partial h}\left(x,y,H(x,y),\frac{\partial H}{\partial x}(x,y),\frac{\partial H}{\partial y}(x,y)\right)-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\left(x,y,H(x,y),\frac{\partial H}{\partial x}(x,y),\frac{\partial H}{\partial y}(x,y)\right)\right)\right.$$
$$\left.-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\left(x,y,H(x,y),\frac{\partial H}{\partial x}(x,y),\frac{\partial H}{\partial y}(x,y)\right)\right)\right)g(x,y)dxdy=0.$$

Great! Let's review exactly what we've proved at this point. We showed that for any function $g\in C^2[\Omega]$ that satisfies the boundary condition of vanishing on $\partial\Omega$, the above integral is equal to zero. Ok, what did we do at this analogous point in the proof of Theorem 1.3.3. We applied the analog of Lemma 2.4.2 to our equation. So, let us do that here as well! Noticing that the above equation satisfies all of the hypothesis of Lemma 2.4.2 we get that Lemma 2.4.2 implies that:

$$\frac{\partial F}{\partial h}\left(x, y, H(x,y), \frac{\partial H}{\partial x}(x,y), \frac{\partial H}{\partial y}(x,y)\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\left(x, y, H(x,y), \frac{\partial H}{\partial x}(x,y), \frac{\partial H}{\partial y}(x,y)\right)\right)$$
$$- \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\left(x, y, H(x,y), \frac{\partial H}{\partial x}(x,y), \frac{\partial H}{\partial y}(x,y)\right)\right) = 0$$

on all of Ω^{int}. Since the left-hand side in the above equation is continuous, it continuously extends to the boundary of Ω and thus the above equation holds on all of Ω (in this argument we are technically using the fact that Ω is the closure of its interior because this then implies that every boundary point of Ω has points where the above equation holds arbitrarily close to it). But now notice that this equation is $H(x,y)$ being plugged into the Euler-Lagrange partial differential equation and so we get that our extremum $H(x,y)$ does indeed satisfy the Euler-Lagrange partial differential equation! With this we have proved the theorem.

■

As before, the following is a really cool definition to make after the proof of the above theorem.

Definition 2.4.7: *Let Ω and J be the region and function respectively as described in the statement of Theorem 2.4.4. Then the **variational derivative**, or **functional derivative**, of the functional J is defined as:*

$$\frac{\delta J}{\delta h}[h] = \frac{\partial F}{\partial h}(x, y, h, h_x, h_y) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}(x, y, h, h_x, h_y)\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}(x, y, h, h_x, h_y)\right).$$

This is the left-hand side of the Euler-Lagrange partial differential equation in Theorem 2.4.4. Notice that the necessary condition proved in the above theorem can now be rewritten in the nice form that a necessary condition for a surface $H(x,y)$ to be an extremum of the functional J is that it satisfies:

$$\frac{\delta J}{\delta h}[H] \equiv 0$$

In other words, the variational derivative of the functional J must be equal to zero at the surface $H(x,y)$.

Isn't the above theorem really cool?! It's a direct generalization of the Euler-Lagrange differential equation that we had in the previous chapter except that this time we are dealing with functionals of the form $J[h(x,y)]$. This will be a crucial theorem to us when we study the variational properties of surfaces since the above theorem allows us to tell what goes on with a quantity that depends on surfaces when we vary them. Contain your excitement because now we're going to prove a version of the above theorem that involves n variables! Then after the proof of following theorem, we will look at an example of an application of these theorems.

In the n-variable version, since we are dealing with possibly more than two variables we can't apply Green's Theorem anymore. We can however instead use the Divergence Theorem (which by the way Green's Theorem is a direct corollary of). We could technically have used the Divergence Theorem in the proof of the above theorem, but the only difference would have been how we wrote down the contour integral in the above proof notation wise.

Theorem 2.4.8 (The Euler-Lagrange Partial Differential Equation for Functionals of $\boldsymbol{n}$-Variable Functions): *Let $\Omega \subseteq \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ that the Divergence Theorem applies to. Suppose that Ω is the closure of its interior and that its boundary $\partial\Omega$ is a set of zero Jordan content. Suppose also that we have a function of the form $f : \partial\Omega \to \mathbb{R}$ and that the set:*

$$\{(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n)) \in \mathbb{R}^{n+1} : (x_1, x_2, \dots, x_n) \in \partial\Omega\}$$

can be parametrized by a non-singular C^2 curve $\gamma(t)$ in $\mathbb{R}^{n+1}$. This will serve as our boundary condition for the functions in the domain of our functional. Now, let J be a functional of the form (in this theorem and proof, let $\int$ denote an integral over a region in $\mathbb{R}^n$):

$$J[h(x_1, x_2, \dots, x_n)] = \int_{\Omega} F(x_1, x_2, \dots, x_n, h, h_{x_1}, h_{x_2}, \dots, h_{x_n}) \prod_{i=1}^{n} dx_i$$

where $F \in C^2[\mathbb{R}^{2n+1}]$. Let J's domain be the set of "hypersurfaces" $h(x_1, x_2, \dots, x_n) \in C^2[\Omega]$ that satisfy the boundary conditions:

$$h(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) \quad if \quad (x_1, x_2, \dots, x_n) \in \partial\Omega$$

(in other words, the h's pass through γ over $\partial\Omega$). Then a necessary condition for the hypersurface $H(x_1, x_2, \dots, x_n)$ to be a local extremum of the functional J is that it satisfies the Euler-Lagrange partial differential equation on Ω:

$$\frac{\partial F}{\partial h}(x_1, x_2, \dots, x_n, h, h_{x_1}, h_{x_2}, \dots, h_{x_n}) - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial h_{x_1}}(x_1, x_2, \dots, x_n, h, h_{x_1}, h_{x_2}, \dots, h_{x_n})\right)$$
$$- \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial h_{x_2}}(x_1, x_2, \dots, x_n, h, h_{x_1}, h_{x_2}, \dots, h_{x_n})\right) - \cdots$$
$$- \frac{\partial}{\partial x_n}\left(\frac{\partial F}{\partial h_{x_n}}(x_1, x_2, \dots, x_n, h, h_{x_1}, h_{x_2}, \dots, h_{x_n})\right) = 0.$$

Omitting the arguments of F, this equation takes the nicer to look at form:

$$\frac{\partial F}{\partial h} - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial h_{x_1}}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial h_{x_2}}\right) - \cdots - \frac{\partial}{\partial x_n}\left(\frac{\partial F}{\partial h_{x_n}}\right) = 0$$

or:

$$\frac{\partial F}{\partial h} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial h_{x_j}}\right) = 0.$$

I just want to point out the fact that if we wanted to use the indexed argument notation, then the above functional and Euler-Lagrange partial differential equation can be respectively written down as:

$$J[h(\{x_k\}_{k=1}^n)] = \int_\Omega F\left(\{x_k\}_{k=1}^n, h, \{h_{x_k}\}_{k=1}^n\right) \prod_{i=1}^n dx_i$$

and

$$\frac{\partial F}{\partial h}\left(\{x_k\}_{k=1}^n, h, \{h_{x_k}\}_{k=1}^n\right) - \sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial h_{x_j}}\left(\{x_k\}_{k=1}^n, h, \{h_{x_k}\}_{k=1}^n\right)\right) = 0.$$

As you can see, the indexed argument notation made these expressions nice, short, and explicit (three things that I like in a mathematical expression). The indexed argument notation helps even more when stating the further generalization of this theorem.

Proof: As you might have suspected, this theorem is proved just like the previous theorem except that there are more variables involved. For this reason, I will be a little bit briefer on some of the details in this proof. Suppose that $H(x_1, x_2, \dots, x_n)$ is an extremum of the functional J. We want to prove that $H(x_1, x_2, \dots, x_n)$ satisfies the above Euler-Lagrange partial differential equation. Take any function $g \in C^2[\Omega]$ that satisfies the boundary condition of vanishing on $\partial\Omega$ and form the linear flow $\Lambda : \Omega \times [-1, 1] \to \mathbb{R}$ defined by:

$$\Lambda(x_1, x_2, \dots, x_n, t) = H(x_1, x_2, \dots, x_n) + g(x_1, x_2, \dots, x_n) \cdot t.$$

Notice again that this Λ is a "linear flow of hypersurfaces" that passes through our extremum $H(x_1, x_2, \dots, x_n)$ at time $t = 0$ (meaning $\Lambda(x_1, x_2, \dots, x_n, 0) = H(x_1, x_2, \dots, x_n)$) and flows with constant speed $g(x_1, x_2, \dots, x_n)$ (meaning $\frac{\partial \Lambda}{\partial t} = g(x_1, x_2, \dots, x_n)$). Now consider the real valued function:

$$\mathcal{G}(t) = J[\Lambda(x_1, x_2, \dots, x_n, t)] =$$

$$\int_\Omega F(x_1, x_2, \dots, x_n, \Lambda(x_1, x_2, \dots, x_n, t), \frac{\partial \Lambda(x_1, x_2, \dots, x_n, t)}{\partial x_1}, \frac{\partial \Lambda(x_1, x_2, \dots, x_n, t)}{\partial x_2},$$

$$\dots, \frac{\partial \Lambda(x_1, x_2, \dots, x_n, t)}{\partial x_n}) \prod_{i=1}^n dx_i.$$

Here all we did was for each time $t \in [-1, 1]$ we plugged in the hypersurface $\Lambda(x_1, x_2, \dots, x_n, t)$ into the functional J. Now as before, we need the following lemma that says that $\frac{d}{dt}\left(\mathcal{G}(t)\right)\Big|_{t=0} = 0$.

Lemma 2.4.9: *Our function $\mathcal{G}(t)$ above is differentiable and its derivative at $t = 0$ is equal to zero:*

$$\mathcal{G}'(0) = \frac{d}{dt}(J[\Lambda(x_1, x_2, \dots, x_n, t)])\Big|_{t=0} =$$

$$\frac{d}{dt}(\int_\Omega F(x_1, x_2, \dots, x_n, \Lambda(x_1, x_2, \dots, x_n, t), \frac{\partial\Lambda(x_1, x_2, \dots, x_n, t)}{\partial x_1}, \frac{\partial\Lambda(x_1, x_2, \dots, x_n, t)}{\partial x_2},$$

$$\dots, \frac{\partial\Lambda(x_1, x_2, \dots, x_n, t)}{\partial x_n})\prod_{i=1}^{n} dx_i)\Big|_{t=0} = 0.$$

Proof: I will leave the proof of this lemma to the reader since its proof goes exactly the same way as the proof of Lemma 2.4.5 in the proof of the previous theorem, except that in certain places just more variables are needed to be written in. I bet that you could even literally copy and paste the proof of Lemma 2.4.5 here and replace all of the (x, y)'s with $(x_1, x_2, \dots, x_n)$'s and (x, y, t)'s with $(x_1, x_2, \dots, x_n, t)$'s to get a valid proof of this lemma.

■

Now we are ready to do the exciting calculation that proves the theorem. You know, I tried to write out the following calculations in normal notation, but the length of the equations got way out of hand. I feel like up to this point I was able to get away with normal notation in this proof. But for the following, I think that it would be madness on my side to try and write out all of the following expressions without the use of some contracted notation! So, let us use the indexed argument notation in the following calculations. In the above lemma we proved that:[15]

$$\frac{d}{dt}\left(\int_\Omega F\left(\{x_k\}_{k=1}^n, \Lambda(\{x_k\}_{k=1}^n, t), \left\{\frac{\partial\Lambda}{\partial x_m}(\{x_k\}_{k=1}^n, t)\right\}_{m=1}^n\right)\prod_{i=1}^n dx_i\right)\Bigg|_{t=0} = 0.$$

Carry the above $\frac{d}{dt}$ derivative under the integral sign (which we can do since the integrand is continuously differentiable and our domain of integration is compact) to get that:

$$\frac{d}{dt}\left(\int_\Omega F\left(\{x_k\}_{k=1}^n, \Lambda(\{x_k\}_{k=1}^n, t), \left\{\frac{\partial\Lambda}{\partial x_m}(\{x_k\}_{k=1}^n, t)\right\}_{m=1}^n\right)\prod_{i=1}^n dx_i\right)\Bigg|_{t=0}$$

$$= \int_\Omega \frac{d}{dt}\left(F\left(\{x_k\}_{k=1}^n, \Lambda(\{x_k\}_{k=1}^n, t), \left\{\frac{\partial\Lambda}{\partial x_m}(\{x_k\}_{k=1}^n, t)\right\}_{m=1}^n\right)\right)\Bigg|_{t=0}\prod_{i=1}^n dx_i$$

[15] I tried to write this proof as analogously as I could to the proof of the previous theorem. As you go through these calculations I recommend at each step finding the analog step in the proof of the previous theorem so that at each step you have a lower dimensional idea of where in the proof we're located.

$$= \int_\Omega \left(\frac{\partial F}{\partial h}\left(\{x_k\}_{k=1}^n, \Lambda(\{x_k\}_{k=1}^n, 0), \left\{ \frac{\partial \Lambda}{\partial x_m}(\{x_k\}_{k=1}^n, 0) \right\}_{m=1}^n \right) \frac{\partial \Lambda(\{x_k\}_{k=1}^n, 0)}{\partial t} - \sum_{j=1}^n \frac{\partial F}{\partial h_{x_j}}\left(\{x_k\}_{k=1}^n, \Lambda(\{x_k\}_{k=1}^n, 0), \left\{ \frac{\partial \Lambda}{\partial x_m}(\{x_k\}_{k=1}^n, 0) \right\}_{m=1}^n \right) \frac{\partial^2 \Lambda(\{x_k\}_{k=1}^n, 0)}{\partial x_j \partial t} \right) \prod_{i=1}^n dx_i = 0.$$

Wow, our equations are getting extremely long if written without the aid of some sort of contracted notation![16] Let me remind you that we're doing very advanced calculus and so it's understandable that our equations might start looking complicated. Now let us plug in the equation that we had for our flow: $\Lambda(x_1, x_2, \dots, x_n, t) = H(x_1, x_2, \dots, x_n) + g(x_1, x_2, \dots, x_n) \cdot t$ into the above equation. We will then get that:

$$\int_\Omega \left(\frac{\partial F}{\partial h}\left(\{x_k\}_{k=1}^n, H(\{x_k\}_{k=1}^n), \left\{ \frac{\partial H}{\partial x_m}(\{x_k\}_{k=1}^n) \right\}_{m=1}^n \right) g(\{x_k\}_{k=1}^n) - \sum_{j=1}^n \frac{\partial F}{\partial h_{x_j}}\left(\{x_k\}_{k=1}^n, H(\{x_k\}_{k=1}^n), \left\{ \frac{\partial H}{\partial x_m}(\{x_k\}_{k=1}^n) \right\}_{m=1}^n \right) \frac{\partial g(\{x_k\}_{k=1}^n)}{\partial x_j} \right) \prod_{i=1}^n dx_i = 0.$$

Let us for the time being stop writing the arguments of the partials of F and g so as to make the equations shorter. Rewriting the above equation but omitting the arguments of the partials of F and g gives:

$$Equation\ 2.4.10: \qquad \int_\Omega \left(\frac{\partial F}{\partial h} \cdot g - \sum_{j=1}^n \frac{\partial F}{\partial h_{x_j}} \cdot \frac{\partial g}{\partial x_j} \right) \prod_{i=1}^n dx_i = 0.$$

Now, let's get rid of the annoying $\partial / \partial x_j$ partials of g. How did we do this analogous step in our proof of Theorem 1.3.3? We integrated by parts. Here we can't integrate by parts, but what we can do is apply the Divergence Theorem. Take the $\int_\Omega \left(\sum_{j=1}^n \frac{\partial F}{\partial h_{x_j}} \cdot \frac{\partial g}{\partial x_j} \right) \prod_{i=1}^n dx_i$ term in the above integral. Applying the identity:

$$u \frac{\partial v}{\partial w} = \frac{\partial}{\partial w}(uv) - \frac{\partial u}{\partial w} v$$

to each $\frac{\partial F}{\partial h_{x_j}} \cdot \frac{dg}{\partial x_j}$ term give us that we can rewrite our term as:

[16] You haven't seen anything yet though!

$$\int_{\Omega} \left(\sum_{j=1}^{n} \frac{\partial F}{\partial h_{x_j}} \cdot \frac{\partial g}{\partial x_j} \right) \prod_{i=1}^{n} dx_i$$

$$= \int_{\Omega} \left(\sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} \left(\frac{\partial F}{\partial h_{x_j}} \cdot g \right) \right) - \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} \left(\frac{\partial F}{\partial h_{x_j}} \right) \cdot g \right) \right) \prod_{i=1}^{n} dx_i.$$

Applying the Divergence Theorem to the $\int_{\Omega} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial F}{\partial h_{x_j}} \cdot g \right) \right) \prod_{i=1}^{n} dx_i$ portion of the integral turns the above quantity into (let $\oiint$ here denote a hypersurface integral):

$$\oiint_{\partial\Omega} \left(\frac{\partial F}{\partial h_{x_1}} \cdot g, \frac{\partial F}{\partial h_{x_2}} \cdot g, \dots, \frac{\partial F}{\partial h_{x_n}} \cdot g \right) \cdot d\vec{n} - \int_{\Omega} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial F}{\partial h_{x_j}} \right) \cdot g \right) \prod_{i=1}^{n} dx_i.$$

where $d\vec{n}$ denotes the differential normal vector to $\partial\Omega$. Since g vanishes on $\partial\Omega$, we get that the above $\oiint$ integral is equal to zero and so we're only left with:

$$- \int_{\Omega} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial F}{\partial h_{x_j}} \right) \cdot g \right) \prod_{i=1}^{n} dx_i.$$

So, by an application of the Divergence Theorem we have proved that our term is equal to:

$$\int_{\Omega} \left(\sum_{j=1}^{n} \frac{\partial F}{\partial h_{x_j}} \cdot \frac{\partial g}{\partial x_j} \right) \prod_{i=1}^{n} dx_i = - \int_{\Omega} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial F}{\partial h_{x_j}} \right) \cdot g \right) \prod_{i=1}^{n} dx_i.$$

We basically "integrated by parts" the integral on the left via the Divergence Theorem. Plugging our term back into Equation 2.4.10 gives:

$$\int_{\Omega} \left(\frac{\partial F}{\partial h} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial F}{\partial h_{x_j}} \right) \right) g(\{x_k\}_{k=1}^{n}) \prod_{i=1}^{n} dx_i = 0.$$

Since the boundary of our region of integration Ω has zero Jordan content, the above integral doesn't get affected if we only integrate over Ω's interior:

$$\int_{\Omega^{int}} \left(\frac{\partial F}{\partial h} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial F}{\partial h_{x_j}} \right) \right) g(\{x_k\}_{k=1}^{n}) \prod_{i=1}^{n} dx_i = 0.$$

We rewrote the above integral in this way because we are about to apply Lemma 2.4.3 to this integral, and technically for this lemma to be applicable we need our region of integration to be open.

Great! Let's review exactly what we've proved at this point. We showed that for any function $g \in C^2[\Omega]$ that satisfies the boundary condition of vanishing on $\partial\Omega$, the above integral is equal to zero. So, noticing that the above equation satisfies all of the hypothesis of Lemma 2.4.3 we get that Lemma 2.4.3 implies that:

$$\frac{\partial F}{\partial h} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial h_{x_j}}\right) = 0.$$

on all of Ω^{int}. Or, if we write out the arguments of F:

$$\frac{\partial F}{\partial h}\left(\{x_k\}_{k=1}^n, H(\{x_k\}_{k=1}^n), \left\{\frac{\partial H}{\partial x_m}(\{x_k\}_{k=1}^n)\right\}_{m=1}^n\right)$$
$$- \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial h_{x_j}}\left(\{x_k\}_{k=1}^n, H(\{x_k\}_{k=1}^n), \left\{\frac{\partial H}{\partial x_m}(\{x_k\}_{k=1}^n)\right\}_{m=1}^n\right)\right) = 0$$

on all of Ω^{int}. Since the left-hand side in the above equation is continuous, it continuously extends to the boundary of Ω and thus the above equation holds on all of Ω (in this argument we are again technically using the fact that Ω is the closure of its interior because this then implies that every boundary point of Ω has points where the above equation holds arbitrarily close to it). But notice that the above equation is just $H(\{x_k\}_{k=1}^n) = H(x_1, x_2, \dots, x_n)$ being plugged into the Euler-Lagrange partial differential equation. So we get that our extremum $H(x_1, x_2, \dots, x_n)$ does indeed satisfy the Euler-Lagrange partial differential equation! Thus we have proved the theorem.

■

Definition 2.4.11: *Let Ω and J be a region and functional as in the statement of Theorem 2.4.8. Then the **variational derivative**, or **functional derivative**, of the functional J is defined as:*

$$\frac{\delta J}{\delta h}[h] = \frac{\partial F}{\partial h}(x_1, x_2, \dots, x_n, h, h_{x_1}, h_{x_2}, \dots, h_{x_n}) - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial h_{x_1}}(x_1, x_2, \dots, x_n, h, h_{x_1}, h_{x_2}, \dots, h_{x_n})\right)$$
$$- \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial h_{x_2}}(x_1, x_2, \dots, x_n, h, h_{x_1}, h_{x_2}, \dots, h_{x_n})\right) - \cdots$$
$$- \frac{\partial}{\partial x_n}\left(\frac{\partial F}{\partial h_{x_2}}(x_1, x_2, \dots, x_n, h, h_{x_1}, h_{x_2}, \dots, h_{x_n})\right).$$

Or in indexed argument notation:

$$\frac{\delta J}{\delta h}[h] = \frac{\partial F}{\partial h}\left(\{x_k\}_{k=1}^n, h, \{h_{x_k}\}_{k=1}^n\right) - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial h_{x_j}}\left(\{x_k\}_{k=1}^n, h, \{h_{x_k}\}_{k=1}^n\right)\right).$$

This is just the left-hand side of the Euler-Lagrange partial differential equation in Theorem 2.4.8. Notice that the necessary condition proved in the above theorem can now be rewritten in the nice form that a necessary condition for a hypersurface $H(x_1, x_2, \dots, x_n)$ to be an extremum of the functional J is that it satisfies:

$$\frac{\delta J}{\delta h}[H] \equiv 0.$$

Now that we have the above two theorems, let's look at an example of their application.

Example 2.4.12: (Minimal surfaces) As an example of an application of the above theorems, let's derive the partial differential equation of a two-dimensional minimal surface. Minimal surfaces are defined as surfaces that minimize surface area while satisfying some given boundary conditions (kind like minimizing curves who minimize arc-length while satisfying the boundary conditions of passing through two given points). Let's set up a problem of this sort. Suppose that Ω is the closed unit disk in $\mathbb{R}^2$:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Suppose also that we have a continuously differentiable parametrized curve $\gamma : [0, 2\pi) \to \mathbb{R}^3$ defined by:

$$\gamma(t) = \big(\cos(t), \sin(t), f(t)\big)$$

where f is some continuously differentiable function. This curve γ will serve as the boundary condition for our minimal surface problem.

Now, the minimal surface problem here is to find a surface S that passes through the curve γ and that has the minimum surface area over the unit disk. A pretty cool problem, and it's easily handled with our variational theory. For simplicity, let us in this example only consider surfaces that are graphs of a function $h(x, y)$ over the unit disk. We consider only such surfaces for now because at the moment we don't really have the definitions or machinery at our disposal to consider this problem in terms of more general types of surfaces (such as those that have other types of parametrizations). In the differential geometry chapters will return and solve this problem in more generality where we will consider more general types of surfaces and boundary conditions. That said, we can now reformulate our problem in terms of our variational theory. Our problem is equivalent to finding the minimum of the following functional

$$J[h(x, y)] = \iint_{\Omega} \sqrt{1 + h_x^2 + h_y^2}\, dx dy$$

(notice that this is the surface area integral) where the domain of this functional is the set of $h \in C^2[\Omega]$ that satisfy the boundary conditions:

$$h(\cos(t), \sin(t)) = f(t).$$

Notice that this boundary condition merely says that the surface generated by the graph of h must intersect the curve γ on the unit circle. Let's find the partial differential equation that h

must satisfy in order to minimize the above surface area functional (or in other words, to be a minimal surface).

Notice that in the problem set up here, all of the hypothesis of Theorem 2.4.4 are satisfied. So, we can apply Theorem 2.4.4 to get that in order for our h to be a minimal surface it must satisfy the following partial differential equation:

$$\frac{\partial F}{\partial h} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right) = 0$$

where,

$$F = \sqrt{1 + h_x^2 + h_y^2}$$

Plugging $\sqrt{1 + h_x^2 + h_y^2}$ into F in the above partial differential equation gives us that h must satisfy (at one point in the following calculation I will use the fact that $h_{xy} = h_{yx}$, which is true since h is twice continuously differentiable):

$$\frac{\partial F}{\partial h} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right)$$

$$= \frac{\partial\left(\sqrt{1 + h_x^2 + h_y^2}\right)}{\partial h} - \frac{\partial}{\partial x}\left(\frac{\partial\left(\sqrt{1 + h_x^2 + h_y^2}\right)}{\partial h_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\left(\sqrt{1 + h_x^2 + h_y^2}\right)}{\partial h_y}\right)$$

$$= 0 - \frac{\partial}{\partial x}\left(\frac{h_x}{\sqrt{1 + h_x^2 + h_y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{h_y}{\sqrt{1 + h_x^2 + h_y^2}}\right)$$

$$= -\frac{h_{xx}}{\sqrt{1 + h_x^2 + h_y^2}} + \frac{h_x\left(h_x h_{xx} + h_y h_{yx}\right)}{\sqrt{1 + h_x^2 + h_y^2}^3} - \frac{h_{yy}}{\sqrt{1 + h_x^2 + h_y^2}} + \frac{h_y\left(h_x h_{xy} + h_y h_{yy}\right)}{\sqrt{1 + h_x^2 + h_y^2}^3}$$

$$= \frac{-h_{xx}\left(1 + h_y^2\right) + 2h_x h_y h_{xy} - h_{yy}(1 + h_x^2)}{\sqrt{1 + h_x^2 + h_y^2}^3} = 0$$

Since the denominator term $\sqrt{1 + h_x^2 + h_y^2}$ is always strictly bigger than 0, we get that the above partial differential equation is in fact equivalent to:

$$h_{xx}\left(1 + h_y^2\right) - 2h_x h_y h_{xy} + h_{yy}(1 + h_x^2) = 0$$

Wow! So we get that a necessary condition for h to be a minimal surface (or in other words, to minimize the above surface area functional) is that it satisfies the above partial differential equation. What a wonderful equation! With this in fact we have derived our first partial differential equation for minimal surfaces! This is a first step towards the variational study of minimal surfaces and we will return to this subject in the differential geometry chapters where we will show that this is a special case of a fact that says that all minimal surfaces have mean curvature zero.

Chapter 3: Variational Problems with Subsidiary Conditions and Cool Applications

“Can thee prove:

$$\int_0^\infty \left(\frac{e^{-x}}{x^2} + \frac{1}{2}\frac{e^{-x}}{x} - \frac{1}{x(e^x - 1)}\right) dx = \ln\left(\frac{\sqrt{2\pi}}{e}\right)$$

– Amonomous” – Anonymous person at the University of Washington

Section 1: The Plan

In this chapter we finally get to my favorite subject in first order variational theory. In this chapter we will be studying how to solve variational problems subjected to certain types of subsidiary conditions. After that I would like to show you a really cool application that variational theory has to the proof of Green’s Theorem and the Divergence Theorem and how the calculus of variations can be used to reformulate Newton’s laws of classical mechanics. The last topic that we will discuss in this chapter is why solutions to Euler-Lagrange differential equations are invariant under diffeomorphic maps. This last subject will be very important to us later on since it will allow us to shorten some of our calculations in the differential geometry chapters by a lot! In fact, some of the calculations in the differential geometry chapters are nearly impossible to carry out without this tool since the original un-shortened versions of these calculations are inhumanly long. We’ll meet them when we get to them.

Section 2: The Isoperimetric Problems

[See future edition of this book]

Section 3: Variational Problems on Surfaces

The second type of subsidiary condition that the domain of a functional can be subjected to is that all of the curves in the functional's domain must lie on a surface. Let's look at the most classical example of such a functional. Suppose that we have surface S sitting in $\mathbb{R}^3$ that is equal to the level set of some continuously differentiable function $g : \mathbb{R}^3 \to \mathbb{R}$. In other words, let:

$$S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = c\}.$$

for some fixed constant c. This type of "level set" representation of a surface is quite natural and is often a very convenient way to represent a surface. In fact, this type of surface representation is more general than the usual representation of a surface as the graph of a function that we're so often used to. In calculus class you were probably introduced to representing surfaces as the graph of a function of the form $z = f(x, y)$. But notice that this is merely a special case of the above type of representation since we can set $g(x, y, z) = z - f(x, y)$ and our surface will now be the set of points $(x, y, z) \in \mathbb{R}^3$ that satisfy the equation $g(x, y, z) = z - f(x, y) = 0$ (in this case $c = 0$). The point that I'm trying to convey here is that the above form of representation of a surface S is quite natural and we will be studying it (along with other types of surface representations) in more depth in the differential geometry chapters.

So again, suppose that our surface S has the above level set surface representation through g. Let's also add the condition that ∇g never vanishes on S. There are two reasons for adding this condition, the first of which is to prevent the surface from behaving "weirdly" in any way. More importantly, this condition allows us to easily talk about a tangent space to the surface since then the tangent plane to the surface at a point p is a plane that intersect p and that is perpendicular to the line given by $l(t) = p + \nabla g(p)t$.[17] Ok, now suppose that (A_x, A_y, A_z) and (B_x, B_y, B_z) are two points on the surface S. Now, let J be a functional of the form:

$$J[u(t), v(t), w(t)] = \int_{t_0}^{t_1} F\big(t, u(t), v(t), w(t), u'(t), v'(t), w'(t)\big)dt$$

where $F \in C^2[\mathbb{R}^7]$ and where J's domain is the set of curves $(u, v, w) \in \prod_{k=1}^{3} C^2[t_0, t_1]$ that satisfy the boundary conditions:

$$\big(u(t_0), v(t_0), w(t_0)\big) = \big(A_x, A_y, A_z\big) \quad and \quad \big(u(t_1), v(t_1), w(t_1)\big) = \big(B_x, B_y, B_z\big)$$

and that lie on the surface S:

[17] The fact that ∇g is orthogonal to the tangent plane of a level set of g is a famous result from calculus. I give a quick proof of this fact as a review after Theorem 4.3.2 in Chapter 4.

$$\forall t \in [t_0, t_1], \qquad g\big(u(t), v(t), w(t)\big) = c.$$

So basically J is a functional whose domain is the set of twice continuously differentiable curves that lie on the surface S and whose endpoints are $\left(A_x, A_y, A_z\right)$ and $\left(B_x, B_y, B_z\right)$. The variational problem that can now be asked is how do you find the extremum of such a functional J. Notice that this problem is much harder than anything that we've dealt with before since now when we vary our curves we must make sure that our curves always stay on the surface (in order to remain in the domain of J of course). In the following theorem we will derive a necessary condition that any extremum curve of such a functional J must satisfy. This will be the ultimate theorem in the calculus of variations portion of this book (Chapters one through three), and it will take some effort to prove it.

But before we do proceed to the rigorous proof of the following theorem, let's discuss the intuitive approach to the problem of finding the extremums of such a functional J. As we will see, it turns out that a necessary condition (after some technical refinement) for a curve $\big(U(t), V(t), W(t)\big)$ in the domain of J to be an extremum of J is that it satisfies the following vector differential equation:

$$\nabla_\delta J[U, V, W] = \lambda(t) \nabla g\big(U(t), V(t), W(t)\big)$$

for some real valued function $\lambda(t)$ (we'll discuss this differential equation in more detail in the theorem below). Some of you might recognize this equation by noticing that it has a striking resemblance to Lagrange's method of finding the extremums of a real-valued multivariable function whose domain is a surface given in a level set representation. Indeed, the two problems are analogous and the ideas behind the approach to both of them are the same.

What is Lagrange's method of finding the extremums of a real valued function $F : \mathbb{R}^3 \to \mathbb{R}$ where the domain of F is given by a surface S represented in the form:

$$S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = c\}?$$

where c is some fixed constant. Lagrange's method states that a necessary condition for a point $p = (x_0, y_0, z_0) \in S$ on the surface S to be a local extremum of F is that it satisfies the following equation:

$$\nabla F(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

for some real number $\lambda \in \mathbb{R}$. In other words, the vector $\nabla F(x_0, y_0, z_0)$ must be linearly dependent to $\nabla g(x_0, y_0, z_0)$. A striking resemblance to the above variational equation indeed! How is this equation usually proved in a calculus course? Indeed one way is to notice that if $\nabla F(x_0, y_0, z_0)$ was not linearly dependent to $\nabla g(x_0, y_0, z_0)$, then if we would move on the surface in and in the opposite direction of the projection of $\nabla F(x_0, y_0, z_0)$ onto the tangent plane to the surface at p, we would get values of F that are both bigger and smaller than $F(x_0, y_0, z_0)$ which would contradict the fact that p is a local extremum of F. However, a more fruitful approach to this problem for our purposes (which is to gain an idea on how to solve the above variational problem) is to consider all of the continuously differentiable curves $\gamma(t) =$

$\big(x(t), y(t), z(t)\big)$ that lie on the surface S and that pass through the point p at time $t = t_0$. Then, since p is a local extremum of F, the fact that $\gamma(t)$ passes through p at time $t = t_0$ implies that $t = t_0$ is a local extremum of the one variable function $F\big(\gamma(t)\big) = F\big(x(t), y(t), z(t)\big)$ (this argument requires the continuity of $\gamma(t)$, which we have since we required that $\gamma(t)$ is continuously differentiable). This means that:

$$\frac{d}{dt}\Big(F\big(x(t), y(t), z(t)\big)\Big)\Big|_{t=t_0} = 0.$$

Expanding the right-hand side gives that:

$$\frac{d}{dt}\Big(F\big(x(t), y(t), z(t)\big)\Big)\Big|_{t=t_0}$$

$$= \frac{\partial F}{\partial x}(x_0, y_0, z_0) \cdot x'(t_0) + \frac{\partial F}{\partial y}(x_0, y_0, z_0) \cdot y'(t_0) + \frac{\partial F}{\partial z}(x_0, y_0, z_0) \cdot z'(t_0)$$

$$= \nabla F(x_0, y_0, z_0) \cdot \gamma'(t_0) = 0.$$

(Remember, $\cdot$ between two vectors always means "vector dot product"). So, we get that $\nabla F(x_0, y_0, z_0)$ must be perpendicular to $\gamma'(t_0)$. Since $\gamma(t)$ can approach the point p from any direction, we get that $\gamma'(t_0)$ can be any vector in the tangent plane to the surface at the point p. Thus the above equation implies that $\nabla F(x_0, y_0, z_0)$ must be perpendicular to the tangent plane to the surface at the point p. Since there is only one line in $\mathbb{R}^3$ that is perpendicular to this tangent plane and $\nabla g(x_0, y_0, z_0)$ is a vector that is perpendicular to this tangent plane at p, we get that we must have that:

$$\nabla F(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

for some real number $\lambda \in \mathbb{R}$. This proves that Lagrange's equation above is indeed a necessary condition for a point to be an extremum of a real-valued multivariable function whose domain is a surface given in a level set representation.

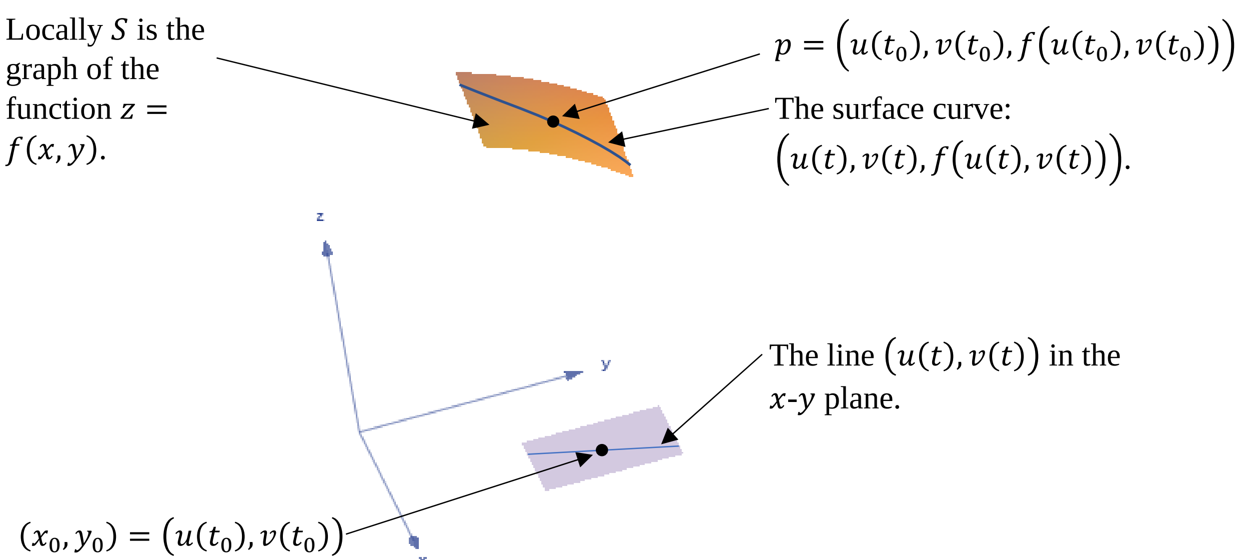
Now let's try to apply these ideas towards solving our variational problem. Take our functional J above and the surface S on which the curves in the domain of J lie upon. The idea to finding the extremums of J is similar to that of F above except that here instead of considering surface curves such as $\gamma(t)$ we will have to take flows of curves on the surface, or "surface flows," in order to say anything about $\nabla_\delta J$. Then we will use the continuity of the surface flows (just like we used the continuity of $\gamma(t)$ above) to show that the derivative of the functional composed with our surface flows at the point in time when our surface flows pass through our extremum is zero. In the case of F this is analogous to the equation $\frac{d}{dt}\Big(F\big(x(t), y(t), z(t)\big)\Big)\Big|_{t=t_0} = 0$. From there, just like with ∇F above, we will be able to derive the above property that $\nabla_\delta J$ always satisfies on an extremum curve.

One of the challenges that we will have to overcome is the ability to construct a surface flow that passes through our extremum "flowing" at any type of speed. The analogous difficulty in the case of F above is: how do you prove the claim that $\gamma(t)$ can approach the point p from any direction? In other words, how do you prove that $\gamma'(t_0)$ can point in any direction in the tangent plane to the surface at the point p. Geometrically speaking, this should be utterly obvious. But there doesn't seem to be any trivial proof of this fact.

One way to prove this fact is to use the implicit function theorem to first argue that locally to p the surface S is the graph of a function of the form $z = f(x, y)$, $x = \tilde{f}(y, z)$, or $y = \tilde{\tilde{f}}(x, z)$. Let's suppose without loss of generality that locally to p, S is the graph of a function $z = f(x, y)$. Then the point p will be given by $\big(x_0, y_0, f(x_0, y_0)\big)$. From here we can take any local line $\big(u(t), v(t)\big)$ that passes through the point (x_0, y_0) at time $t = t_0$ and the curve $\Big(u(t), v(t), f\big(u(t), v(t)\big)\Big)$ will be a curve that lies on the surface S and that passes through the point p at time $t = t_0$ in the direction:

$$\left(u'(t), v'(t), \frac{\partial f}{\partial x}(x_0, y_0) \cdot u'(t_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot v'(t_0)\right).$$

It's not hard to prove that by picking the right line $\big(u(t), v(t)\big)$ that passes through the point (x_0, y_0) at time $t = t_0$ you can get the above direction vector to point in any direction in the tangent plane to the surface at the point p, thus proving the above claim.



To prove the similar sort of thing in the variational problem (that there exists a surface flow that flows with any type of speed through the extremum curve), we will also have to resort to the same trick of applying the implicit function theorem locally to any point on the extremum curve to argue that locally to that point the surface S is the graph a function. From there we will be able to construct a large enough class of surface flows that pass through our extremum so that we will be able to tell what property $\nabla_\delta J$ always satisfies on an extremum curve. Let's begin.

Theorem 3.3.1 (Surface Euler-Lagrange Vector Differential Equation): *Suppose that we have a surface S that is the level set of some continuously differentiable function $g : \mathbb{R}^3 \to \mathbb{R}$. In other words:*

$$S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = c\}.$$

where c is some fixed constant. Suppose also that ∇g never vanishes on S. Let (A_x, A_y, A_z) and (B_x, B_y, B_z) be two points on the surface S. Now, let J be a functional of the form:

$$J[u(t), v(t), w(t)] = \int_{t_0}^{t_1} F\big(t, u(t), v(t), w(t), u'(t), v'(t), w'(t)\big)dt$$

where $F \in C^2[\mathbb{R}^7]$ and where J's domain is the set of curves $(u, v, w) \in \prod_{k=1}^{3} C^2[t_0, t_1]$ that satisfy the boundary conditions:

$$\big(u(t_0), v(t_0), w(t_0)\big) = \big(A_x, A_y, A_z\big) \quad and \quad \big(u(t_1), v(t_1), w(t_1)\big) = \big(B_x, B_y, B_z\big)$$

and that lie on the surface S:

$$\forall t \in [t_0, t_1], \qquad g\big(u(t), v(t), w(t)\big) = c.$$

Now suppose that the curve $\big(U(t), V(t), W(t)\big)$ is a local extremum of J. Then this local extremum curve must satisfy the equation:

$$\nabla_\delta J[U, V, W] = \lambda(t)\nabla g\big(U(t), V(t), W(t)\big)$$

on $t \in (t_0, t_1)$ for some real valued function $\lambda(t)$. If we don't want to write out some of the arguments in the above equation, this equation can be rewritten in the nicer to look at form:

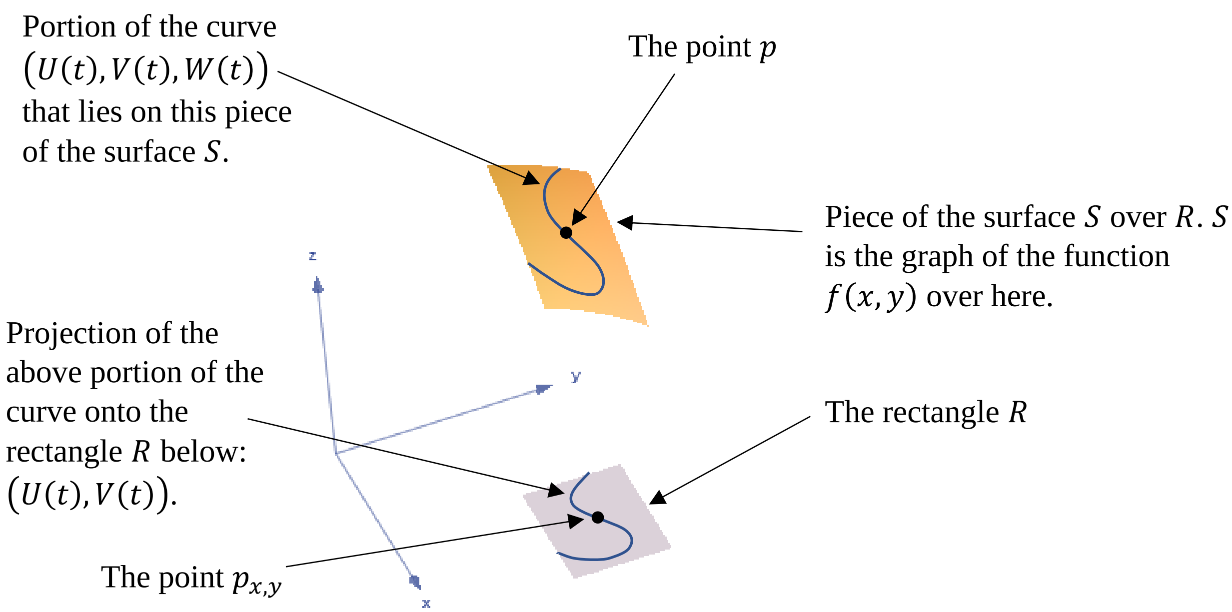
$$\nabla_\delta J = \lambda(t)\nabla g.$$

Proof: The idea of this proof will be to look at the surface S locally as the graph of a function; which we can do by the implicit function theorem. Then, we will take a portion of the curve that is on our local piece of the surface and project it onto one of the fundamental x-y, y-z, or x-z planes and from their treat it as a variational problem of the sort that we dealt with in the previous chapters. Ok, let's begin.

Take any point $t_2 \in (t_0, t_1)$. We will show that the equation in the conclusion of this theorem holds at this arbitrarily chosen point t_2. Before we get to the exciting calculus of variations calculation that proves the theorem, we need to do some technical set up before we get there. Let's call the point $\big(U(t_2), V(t_2), W(t_2)\big)$ that lies on the surface p. Let $p_{x,y}$ denote the projection of p onto the x-y plane. Since ∇g never vanishes on S (a condition that we required), one of the partials $\frac{\partial g}{\partial x}(p), \frac{\partial g}{\partial y}(p), \frac{\partial g}{\partial z}(p)$ is not zero. Let's suppose without loss of generality that the partial $\frac{\partial g}{\partial z}(p) \neq 0$ (the cases when this partial is zero and you have to choose one of the other two partials to get a nonzero partial of g are handled similarly). By the implicit function theorem, since $\frac{dg}{\partial z}(p) \neq 0$ we know that there exists some small compact rectangle $R = [a, b] \times [c, d]$ centered at $p_{x,y}$ in the x-y plane such that the surface S (or more accurately: the level set $\{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = c\}$) can be represented as the graph of a continuously differentiable function over R:

$$z = f(x, y).$$

Basically what we did here is we used the implicit function theorem to solve for z in the equation $g(x, y, z) = 0$ locally to p.[18] So locally to p, S looks like the graph of the function $z = f(x, y)$ over R.



Let us take a closer look at the rectangle R from above:

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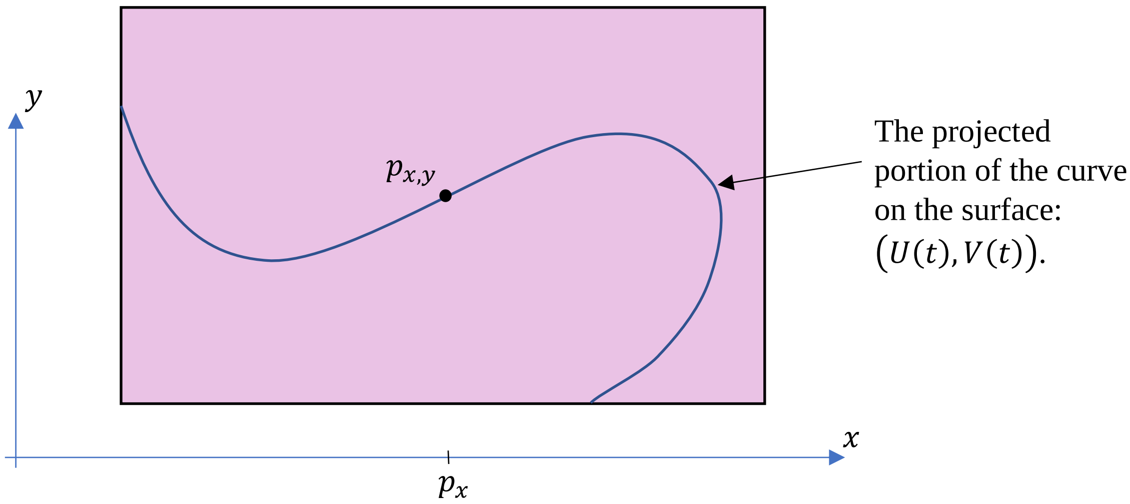
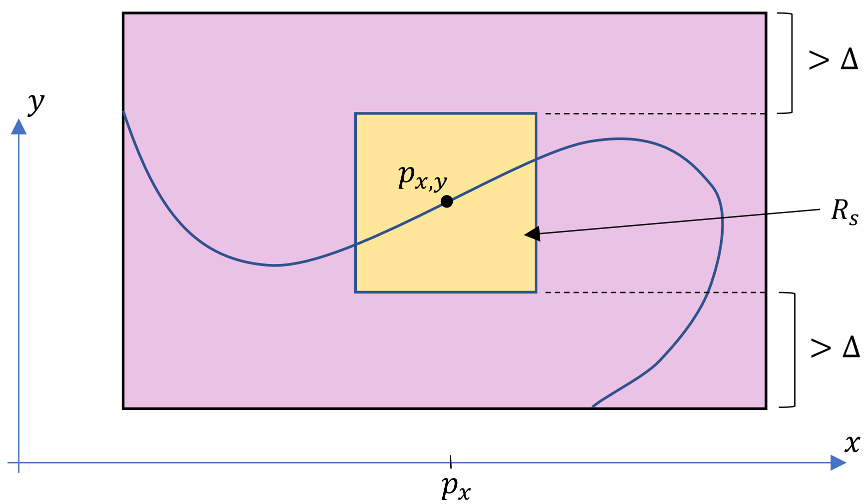
[18] Usually the implicit function theorem is stated in the above form but where the region R is an open ball or open rectangle. However, we can still guarantee that such a compact rectangle exists because both open balls and open rectangles contain a small compact rectangle inside of them. So such a compact rectangle R does exists. We will need the compactness of R later on to produce a certain important inequality.

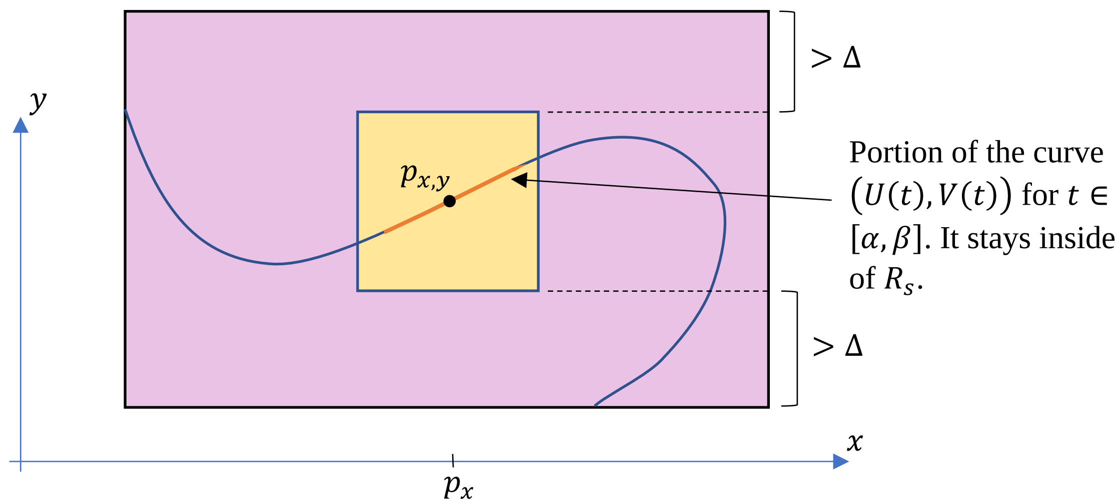


Let p_x denote the projection of $p_{x,y}$ onto the x-axis. Now, let us take any small rectangle R_s contained in R such that the vertical distances between the top and bottom edges of R_s and R are bigger than some positive number $\Delta > 0$.



The subscript s in R_s stands for "small rectangle." You might later wonder why we need the small rectangle R_s. We will need it to create a neighborhood around the projection of our local extremum curve that lies in R so that we can follow the variational theory that we developed in the past two chapters in this proof. We'll discuss this in more detail soon.

There is some time interval $[\alpha, \beta] \subseteq [t_0, t_1]$ centered at t_2 such that the projected curve $(U(t), V(t))$ stays inside of the small rectangle R_s as long as $t \in [\alpha, \beta]$:



Now let's get to some calculus of variations! One idea that might come across right now is that since S is locally the graph of the function $f(x, y)$, any curve $\big(u(t), v(t), w(t)\big)$ that lies on the surface S locally to p must satisfy the equation $w(t) = f\big(u(t), v(t)\big)$. So locally to p the integrand of the integral that defines J can be rewritten as:

$$F\left(t, u(t), v(t), f\big(u(t), v(t)\big), u'(t), v'(t), \frac{d}{dt}\Big(f\big(u(t), v(t)\big)\Big)\right)$$

Since here $u(t)$ and $v(t)$ are functional variables that locally aren't constrained by any kind of subsidiary conditions, we could locally look at this problem as the sort of problem treated in Theorem 2.2.5. This is a valid approach, and if done carefully furnishes a nice proof of this theorem. However, there are some technicalities that you will need to overcome. For example, the proof of Theorem 2.2.5 fundamentally used the fact that every possible extremum curve in the functional's domain had a C^0 open neighborhood around it. This is in fact the reason why we created the above rectangle R_s with the property that the vertical distances between the top and bottom edges of R_s and R are bigger than $\Delta > 0$. R_s is there to create this local C^0 neighborhood around our the projection of our local extremum curve. Another technicality that you will have to overcome is that you will have to prove that the necessary condition for the whole curve $\big(U(t), V(t), W(t)\big)$ to be a local extremum will correspond with the local necessary condition for your projected curve $\big(U(t), V(t)\big)$ to be a local extremum of J. Although to some this might sound obvious, it needs to be carefully proven. If you didn't really understand this then that's all right since we are going to take a different approach.

We will take a more fundamental approach where will prove this theorem from basic variational principles, just like we did with the other Euler-Lagrange Theorems: Theorem 1.3.3, Theorem 2.4.4, and Theorem 2.4.8. We will still have to overcome the above technicalities, but we will get to the same result with almost the same amount of effort as the above proposed proof. Realistically speaking, the main difference between the above approach and the one that we are about to take is that with the above approach it might be possible to skip a portion of the very last calculation in this proof.

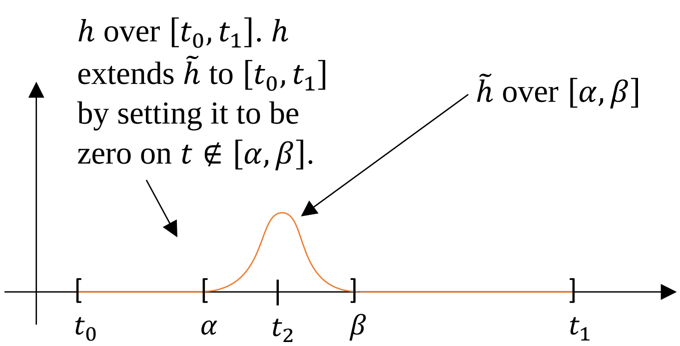
The rest of this proof will follow the model of what we did in the proof of Theorem 1.3.3, except that we have some additional technicalities that we have to overcome before we can get to the exciting calculation that proves the theorem. Let us take any function $\tilde{h} \in C^2[\alpha, \beta]$ that satisfies the boundary conditions:

$$\tilde{h}(\alpha) = \tilde{h}'(\alpha) = \tilde{h}''(\alpha) = 0 \quad and \quad \tilde{h}(\beta) = \tilde{h}'(\beta) = \tilde{h}''(\beta) = 0.$$

Let us "extend" $\tilde{h}$ to the whole time interval $[t_0, t_1]$ by defining h as follows:

$Equation$ 3.3.2:
$$h(t) = \begin{cases} \tilde{h}(t) & if \quad t \in [\alpha, \beta] \\ 0 & if \quad t \notin [\alpha, \beta] \end{cases}.$$

Notice that $h \in C^2[t_0, t_1]$ because $\tilde{h}$ satisfies the above boundary conditions.



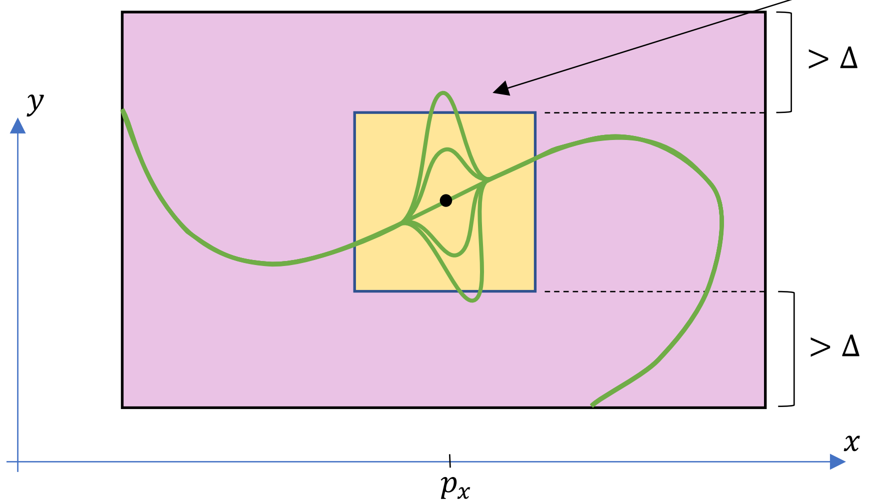
Now, let us form the 2-smooth linear flow $\Lambda : [t_0, t_1] \times [T_0, T_1] \to \mathbb{R}$ (we will soon define T_0 and T_1) by:

$$\Lambda(t, T) = V(t) + h(t)T.$$

This is a flow that passes through $V(t)$ at $T = 0$ and flows with the constant speed $h(t)$. A few remarks are in order. t here is the time variable for the parametrization of the curve $\big(U(t), V(t), W(t)\big)$ while T is the "flow time" for Λ. I usually used the letter t for the "flow times" of our linear flows (as in the previous chapters), but since I am already using t for another purpose, I am going to use capital T here instead for the purpose of representing this flow time.

Also, I said that "we will soon define T_0 and T_1" because we will want to find such suitable numbers that satisfy the following soon-to-be-stated properties. Before we state what these properties are, let us form the "2-dimensional flow" $\big(U(t), \Lambda(t, T)\big)$. 2-dimensional flows are just like any other flow except that here for every fixed T we think of $\big(U(t), \Lambda(t, T)\big)$ as a parametrized 2-dimensional curve of t in the x-y plane. Thus as T varies across the interval $[T_0, T_1]$, we have that $\big(U(t), \Lambda(t, T)\big)$ is a "flow" of 2-dimensional parametrized curves (see the image below).

Now what is the condition that we want T_0 and T_1 to satisfy? We want to find an interval $[T_0, T_1]$ such that this 2-dimensional flow passes through the curve $\big(U(t), V(t)\big)$ at time $T = 0$ and such that for on $t \in [\alpha, \beta]$, the curve $\big(U(t), \Lambda(t,T)\big)$ always stays inside of R for any flow time $T \in [T_0, T_1]$.



The portion of the 2-dimensional flow $\big(U(t), \Lambda(t,T)\big)$ in R at times $T = T_0$, $T = T_0/2$, $T = 0$, $T = T_1/2$, $T = T_1$ (bottom to top). At $T = 0$, this 2-dimensional flow passes through the curve $\big(U(t), V(t)\big)$ and on $t \in [\alpha, \beta]$ this flow always stays inside of R.

Let's find such suitable times T_0 and T_1 that satisfy the above two properties. In order for Λ to pass through the curve $V(t)$ at time $T = 0$ we need $0 \in [T_0, T_1]$. So we must choose $T_0 < 0$ and $T_1 > 0$. Now, we also want to make sure that on $t \in [\alpha, \beta]$, the curve $\big(U(t), \Lambda(t,T)\big)$ always stays inside of R as $T \in [T_0, T_1]$. We will find T_0 and T_1 that do this by using the continuity of the flow Λ. By Theorem 1.2.12, we know that there exists some positive number $\gamma > 0$ such that for $\forall T : |T - 0| = |T| < \gamma$,

$$\left\|\widetilde{\Lambda}(T) - \widetilde{\Lambda}(0)\right\| = \left\|\widetilde{\Lambda}(T) - V(t)\right\| < \Delta.$$

This implies that (you may want to relook at the proof of Theorem 1.2.12 to see why the last equality in the following equation holds),

$$\max_{t\in[t_0,t_1]}\{|\Lambda(t,T) - V(t)|\}$$

$$\leq \max_{t\in[t_0,t_1]}\{|\Lambda(t,T) - V(t)|\} + \max_{t\in[t_0,t_1]}\left\{\left|\frac{\partial\Lambda(t,T)}{\partial t} - V'(t)\right|\right\} + \max_{t\in[t_0,t_1]}\left\{\left|\frac{\partial^2\Lambda(t,T)}{\partial t^2} - V(t)\right|\right\}$$

$$= \sum_{k=0}^{2}\max_{t\in[t_0,t_1]}\left\{\left|\frac{\partial^k\Lambda(t,T)}{\partial t^k} - V^{(k)}(t)\right|\right\} = \left\|\widetilde{\Lambda}(T) - V(t)\right\| < \Delta.$$

And so, we get that for $\forall T : |T| < \gamma$,

$$\max_{t\in[t_0,t_1]}\{|\Lambda(t,T) - V(t)|\} < \Delta.$$

Since $\big(U(t), V(t)\big)$ is contained in R_s for $\forall t \in [\alpha, \beta]$ and the vertical distance between the top and bottom edges of R_s and R are bigger that Δ, get that the above equation implies that on $t \in$

$[\alpha, \beta]$ and for $\forall T : |T| < \gamma$ the curve $\big(U(t), \Lambda(t,T)\big)$ will always lie inside of R. So, we can set $T_0 = -\gamma$ and $T_1 = \gamma$ and we will get the $T \in [T_0, T_1]$ interval that we want. With this we've formed the linear flow Λ and its corresponding 2-dimensional flow $\big(U(t), \Lambda(t,T)\big)$ that satisfy the above two properties.

Notice that since $h(t) = 0$ on $t \notin [\alpha, \beta]$, our flow $\Lambda(t,T) = V(t) + h(t)T$ can be rewritten in case bracket notation as:

$$\Lambda(t,T) = \begin{cases} V(t) + h(t)T & for \quad t \in [\alpha, \beta] \\ V(t) & for \quad t \notin [\alpha, \beta] \end{cases}$$

Or since $h(t) = \tilde{h}(t)$ on $t \in [\alpha, \beta]$ (see Equation 3.3.2), we get that we can rewrite the above equation as:

$Equation$ 3.3.3:
$$\Lambda(t,T) = \begin{cases} V(t) + \tilde{h}(t)T & for \quad t \in [\alpha, \beta] \\ V(t) & for \quad t \notin [\alpha, \beta] \end{cases}$$

The reason why I wrote Λ in this case bracket notation is that it's sometimes easier to understand how such linear flows look like when they're written out in this form. This is especially so in the case when the only change happening on the curve $\Lambda(t,T)$ as T runs over the interval $[T_0, T_1]$ is in the $t \in [\alpha, \beta]$ section of the curve (notice that on $t \notin [\alpha, \beta]$, $\Lambda(t,T)$ is constantly $V(t)$ and thus nothing is changing there as the flow time T varies).

Now that we have constructed this linear flow Λ that passes through $V(t)$ at $T = 0$ and that always stays inside of R, what do we do with it? Let's use this flow to construct a type of "surface curve flow" $\big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)$ (or "flow of curves that lie on the surface S") that goes through our local extremum $\big(U(t), V(t), W(t)\big)$ at $T = 0$.

Our surface S is the graph of the function $f(x,y)$ locally at p and since $\big(U(t), V(t), W(t)\big)$ is a curve that lies on the surface S, the fact that $\big(U(t), V(t)\big) \in R$ for $\forall t \in [\alpha, \beta]$ implies that for $\forall t \in [\alpha, \beta]$,

$$W(t) = f\big(U(t), V(t)\big).$$

In other words, locally at p (or more precisely: on $t \in [\alpha, \beta]$) we can solve for $W(t)$ in terms of $U(t)$ and $V(t)$. Now let us construct the following function:

$Equation$ 3.3.4:
$$W_\Lambda(t,T) = \begin{cases} f\big(U(t), \Lambda(t,T)\big) & for \quad t \in [\alpha, \beta] \\ W(t) & for \quad t \notin [\alpha, \beta] \end{cases}.$$

The purpose of this $W_\Lambda(t,T)$ function is that, as we will show, for each flow time $T \in [T_0, T_1]$ the following curve lies on the surface S:

$$\forall t \in [t_0, t_1], \qquad \big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big) \in S.$$

Let's show that this is indeed so. Notice that for any flow time $T \in [T_0, T_1]$:

$$\forall t \in [\alpha, \beta], \qquad W_\Lambda(t,T) = f\big(U(t), \Lambda(t,T)\big), \quad so \quad \big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big) \in S,$$

$$\forall t \notin [\alpha, \beta], \qquad \big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big) = \big(U(t), V(t), W(t)\big) \in S.$$

(You may want to look at Equations 3.3.3 and 3.3.4 to see why the second equality holds). So both of the above cases show that for any flow time $T \in [T_0, T_1]$,

$$\forall t \in [t_0, t_1], \qquad \big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big) \in S.$$

Thus, for every flow time $T \in [T_0, T_1]$ the above curve indeed lies on the surface S. So as T varies across the flow time interval $[T_0, T_1]$, we can think of $\big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)$ as a flow of "surface curves" that lie on the surface S (see the image below). And this "surface curve flow", or "surface flow", passes through our local extremum curve $\big(U(t), V(t), W(t)\big)$ at $T = 0$ since if we plug in $T = 0$,

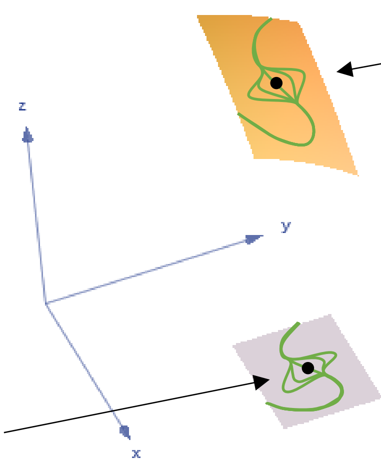
$$\forall t \in [\alpha, \beta], \qquad \big(U(t), \Lambda(t,0), W_\Lambda(t,0)\big) = \Big(U(t), V(t), f\big(U(t), V(t)\big)\Big) = \big(U(t), V(t), W(t)\big),$$

$$\forall t \notin [\alpha, \beta], \qquad \big(U(t), \Lambda(t,0), W_\Lambda(t,0)\big) = \big(U(t), V(t), W(t)\big).$$

And thus,

$$\forall t \in [t_0, t_1], \qquad \big(U(t), \Lambda(t,0), W_\Lambda(t,0)\big) = \big(U(t), V(t), W(t)\big).$$

So our surface flow $\big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)$ does indeed pass through our local extremum curve $\big(U(t), V(t), W(t)\big)$ at $T = 0$.



These green curves represent the portion of the surface flow $\big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)$ that lies on this small piece of the surface at flow times $T = T_0$, $T = T_0/2$, $T = 0$, $T = T_1/2$, $T = T_1$ (left to right). Notice that this surface flow passes through our local extremum curve $\big(U(t), V(t), W(t)\big)$ at flow time $T = 0$.

These green curves represent the portion of the 2-dimensional flow $\big(U(t), \Lambda(t,T)\big)$ that lies in R at flow times $T = T_0$, $T = T_0/2$, $T = 0$, $T = T_1/2$, $T = T_1$ (left to right). Notice that this flow passes through the curve $\big(U(t), V(t)\big)$ at flow time $T = 0$.

Let's plug our surface flow $\big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)$ into J now:

$$\mathcal{G}(T) = J[U(t), \Lambda(t,T), W_\Lambda(t,T)].$$

All we did here was for every flow time $T \in [T_0, T_1]$ we plugged in the surface curve $\big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)$ into J. Now, like in our proof of the other Euler-Lagrange theorems, we should observe that since the surface flow $\big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)$ passes through our local extremum $\big(U(t), V(t), W(t)\big)$ at $T = 0$ and $\big(U(t), V(t), W(t)\big)$ is a local extremum of J, $T = 0$ should be a local extremum of the function $\mathcal{G}(T)$. So we should have that $G'(0) = 0$. Let's prove this rigorously in the following lemma.

Lemma 3.3.5: *Our function $\mathcal{G}(T)$ above is differentiable and its derivative at $T = 0$ is equal to zero:*

$$\mathcal{G}'(0) = \frac{d}{dT}\big(J[U(t), \Lambda(t,T), W_\Lambda(t,T)]\big)\Big|_{T=0}$$

$$= \frac{d}{dT}\left(\int_{t_0}^{t_1} F\left(t, U(t), \Lambda(t,T), W_\Lambda(t,T), U'(t), \frac{\partial \Lambda}{\partial t}(t,T), \frac{\partial W_\Lambda}{\partial t}(t,T)\right) dt\right)\Bigg|_{T=0} = 0.$$

Proof: This lemma is proved similarly to how we always proved the Euler-Lagrange differential equation lemmas before. However, here we're going to have to do a little bit more work because we don't have a ready theorem that says that the surface flows flow continuously with respect to the curve norm. But we still immediately have that the function $\mathcal{G}(T)$ is differentiable because we can carry the flow time derivative under the integral sign (since the above integrand is continuously differentiable and our domain of integration is compact). And in order to prove that the above derivative is equal to zero, it is sufficient to just show that $t = 0$ is a local extremum of our function $\mathcal{G}(t)$. The fact that $\mathcal{G}'(0) = 0$ will then follow from the well-known fact that the derivative of a function at an extremum is equal to zero.

Let's suppose that $\big(U(t), V(t), W(t)\big)$ is a local minimum of the functional J. The proof of this lemma in the case of when it is a local maximum of the functional J is similar. Then by definition there exists a $\delta > 0$ such that for any surface curve $\big(\mathcal{U}(t), \mathcal{V}(t), \mathcal{W}(t)\big) \in \mathrm{dom}(J)$ such that $\big\|\big(\mathcal{U}(t), \mathcal{V}(t), \mathcal{W}(t)\big) - \big(U(t), V(t), W(t)\big)\big\| \le \delta$,

$$J[\mathcal{U}(t), \mathcal{V}(t), \mathcal{W}(t)] \ge J[U(t), V(t), W(t)].$$

Now we deviate a little bit from how we proved this lemma in the other Euler-Lagrange differential equation theorems because here we don't have a ready theorem that says that surface flows flow continuously. We have to directly prove that there exists a small flow time interval $T \in [-\Delta, \Delta]$, where $\Delta > 0$, such that for T in this interval the surface flow $\big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)$ flows at most a distance of δ away from our extremum curve $\big(U(t), V(t), W(t)\big)$ in the curve norm. So let's prove the existence of such a number $\Delta > 0$.

Since $\widetilde{\Lambda}(T)$ is a continuous function (see Theorem 1.2.12), we get that there exists some number $\eta > 0$ such that (in the context of Theorem 1.2.12, here $\varepsilon = \delta$):

$$\forall T \in (-\eta, \eta), \qquad \big\|\widetilde{\Lambda}(T) - \widetilde{\Lambda}(0)\big\| = \|\Lambda(t,T) - V(t)\| \le \delta.$$

So within the flow time interval $T \in (-\eta, \eta)$, the flow $\Lambda(t,T)$ flows a distance of no farther than δ from $V(t)$. Let's see what this tells us about how far the surface flow $\big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)$ flows from our local extremum curve $\big(U(t), V(t), W(t)\big)$ in the flow time interval $T \in (-\eta, \eta)$. We have that for $\forall T \in (-\eta, \eta)$,

$$\big\|\big(U(t), V(t), W(t)\big) - \big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)\big\|$$

$$= \max\{\|U(t) - U(t)\|, \|V(t) - \Lambda(t,T)\|, \|W(t) - W_\Lambda(t,T)\|\}.$$

The norm on the left-hand side is the norm as defined in Definition 2.2.3 and the norms on the right-hand side are the $C^2[t_0, t_1]$ norms as defined in Definition 1.2.3. Since $\|U(t) - U(t)\| = 0$, we get that the above equation can be rewritten as:

$$\big\|\big(U(t), V(t), W(t)\big) - \big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)\big\|$$

$$= \max\{\|V(t) - \Lambda(t,T)\|, \|W(t) - W_\Lambda(t,T)\|\}.$$

Applying Definition 1.2.3 to the norms on the right-hand side gives:

$Equation$ 3.3.6: $$\big\|\big(U(t), V(t), W(t)\big) - \big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)\big\|$$

$$= \max\left\{\max_{t\in[t_0,t_1]}\{|V(t) - \Lambda(t,T)|\}, \max_{t\in[t_0,t_1]}\{|W(t) - W_\Lambda(t,T)|\}\right\}.$$

Since $\Lambda(t,T) = V(t)$ and $W_\Lambda(t,T) = W(t)$ on $t \notin [\alpha, \beta]$ (see Equations 3.3.3 and 3.3.4), the above equation can further be rewritten as:

$$\big\|\big(U(t), V(t), W(t)\big) - \big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)\big\|$$

$$= \max\left\{\max_{t\in[\alpha,\beta]}\{|V(t) - \Lambda(t,T)|\}, \max_{t\in[\alpha,\beta]}\{|W(t) - W_\Lambda(t,T)|\}\right\}$$

because on $t \notin [\alpha, \beta]$, $|V(t) - \Lambda(t,T)|$ and $|W(t) - W_\Lambda(t,T)|$ are zero anyways (in this step all I did was I reduced the inner "max domains" from $t \in [t_0, t_1]$ to $t \in [\alpha, \beta]$). Notice that for $\forall t \in [\alpha, \beta]$ the quantity,

$$|W(t) - W_\Lambda(t,T)| = \big|f\big(U(t), V(t)\big) - f\big(U(t), \Lambda(t,T)\big)\big|.$$

By the intermediate value theorem, we get that there exists some $c \in [0, 1]$ such that:

$$\big|f\big(U(t), V(t)\big) - f\big(U(t), \Lambda(t,T)\big)\big|$$

$$= \left|\frac{\partial f}{\partial y}\Big(U(t), \Lambda(t,T) + c\big(V(t) - \Lambda(t,T)\big)\Big) \cdot \big(V(t) - \Lambda(t,T)\big)\right|$$

$$\leq \max_{(x,y)\in R}\left\{\left|\frac{\partial f}{\partial y}(x,y)\right|\right\}|V(t) - \Lambda(t,T)|.$$

The last $\leq$ comes from the fact that:

$$\Big(U(t), \Lambda(t,T) + c\big(V(t) - \Lambda(t,T)\big)\Big) \in R$$

The quantity $\max_{(x,y)\in R}\left\{\left|\frac{\partial f}{\partial y}(x,y)\right|\right\}$ exists since R is compact and $\frac{\partial f}{\partial y}(x,y)$ is continuous on R. $\frac{\partial f}{\partial y}(x,y)$ is continuous on R because of the implicit function theorem and the fact that g is C^1. So, all of this gives us that:

$$|W(t) - W_\Lambda(t,T)| \le \max_{(x,y)\in R}\left\{\left|\frac{\partial f}{\partial y}(x,y)\right|\right\}|V(t) - \Lambda(t,T)|.$$

Plugging this result into Equation 3.3.6 gives us that for $\forall T \in (-\eta, \eta)$:

$$\left\|\big(U(t), V(t), W(t)\big) - \big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)\right\|$$

$$\le \max\left\{\max_{t\in[t_0,t_1]}\{|V(t) - \Lambda(t,T)|\}, \max_{t\in[t_0,t_1]}\left\{\max_{(x,y)\in R}\left\{\left|\frac{\partial f}{\partial y}(x,y)\right|\right\}|V(t) - \Lambda(t,T)|\right\}\right\}$$

Pulling out the $\max_{(x,y)\in R}\left\{\left|\frac{\partial f}{\partial y}(x,y)\right|\right\}$ out of the second max gives that:

$$\left\|\big(U(t), V(t), W(t)\big) - \big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)\right\|$$

$$\le \max\left\{\max_{t\in[t_0,t_1]}\{|V(t) - \Lambda(t,T)|\}, \max_{(x,y)\in R}\left\{\left|\frac{\partial f}{\partial y}(x,y)\right|\right\}\max_{t\in[t_0,t_1]}\{|V(t) - \Lambda(t,T)|\}\right\}$$

Pulling out the $\max_{t\in[t_0,t_1]}\{|V(t) - \Lambda(t,T)|\}$'s gives:

$$\left\|\big(U(t), V(t), W(t)\big) - \big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)\right\|$$

$$\le \max\left\{1, \max_{(x,y)\in R}\left\{\left|\frac{\partial f}{\partial y}(x,y)\right|\right\}\right\}\max_{t\in[t_0,t_1]}\{|V(t) - \Lambda(t,T)|\}.$$

Since $\max_{t\in[t_0,t_1]}\{|V(t) - \Lambda(t,T)|\} \le \left\|\widetilde{\Lambda}(T) - V(t)\right\|$,[19] we get that the above equation implies that for $\forall T \in (-\eta, \eta)$:

$$\left\|\big(U(t), V(t), W(t)\big) - \big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)\right\| \le \max\left\{1, \max_{(x,y)\in R}\left\{\left|\frac{\partial f}{\partial y}(x,y)\right|\right\}\right\}\delta.$$

So within the flow time interval $T \in (-\eta, \eta)$, the surface flow $\big(U(t), \Lambda(t,T), W_\Lambda(t,T)\big)$ flows from our local extremum curve $\big(U(t), V(t), W(t)\big)$ no more than a distance of

[19] $\max_{t\in[t_0,t_1]}\{|V(t) - \Lambda(t,T)|\} \le \left\|\widetilde{\Lambda}(T) - V(t)\right\|$ because $\left\|\widetilde{\Lambda}(T) - V(t)\right\| = \sum_{k=0}^{2}\max_{t\in[t_0,t_1]}\left\{\left|\frac{\partial^k \Lambda(t,T)}{\partial t^k} - V^{(k)}(t)\right|\right\}$ and this Σ sum of non-negative terms include the term $\max_{t\in[t_0,t_1]}\{|V(t) - \Lambda(t,T)|\}$) itself. We actually used this inequality before.

$\max\left\{1, \max_{(x,y)\in R}\left\{\left|\frac{\partial f}{\partial y}(x,y)\right|\right\}\right\}\delta$. O darn! I wanted the above quantity to be less than or equal to δ, not $\max\left\{1, \max_{(x,y)\in R}\left\{\left|\frac{\partial f}{\partial y}(x,y)\right|\right\}\right\}\delta$. But notice that if we instead chose $\eta > 0$ earlier so that:

$$\forall T \in (-\eta,\eta), \qquad \|\Lambda(t,T) - V(t)\| \leq \delta / \max\left\{1, \max_{(x,y)\in R}\left\{\left|\frac{\partial f}{\partial y}(x,y)\right|\right\}\right\}$$

instead, then the above math would then work out to give us that for $\forall T \in (-\eta,\eta)$,

$$\left\|\big(U(t),V(t),W(t)\big) - \big(U(t),\Lambda(t,T),W_\Lambda(t,T)\big)\right\| \leq \delta.$$

So let's pretend that we chose this $\eta > 0$ instead. Going back to the definition of δ in the beginning of this proof, we get that the above equation implies that for any flow time $T \in (-\eta,\eta)$,

$$J[U(t),\Lambda(t,T),W_\Lambda(t,T)] \geq J[U(t),V(t),W(t)].$$

Or in other words: for any $T \in [-\eta,\eta]$,

$$\mathcal{G}(T) \geq \mathcal{G}(0).$$

So $T = 0$ is a local extremum of $\mathcal{G}(T)$ and thus $\mathcal{G}'(0) = 0$.

■

The above lemma now gives us the power to reformulate the extremum condition of our extremum curve $\big(U(t),V(t),W(t)\big)$ into an extremum condition for the one variable function $\mathcal{G}(T)$ at $T = 0$. In the above lemma we proved that:

$$\frac{d}{dT}\left(\int_{t_0}^{t_1} F\left(t,U(t),\Lambda(t,T),W_\Lambda(t,T),U'(t),\frac{\partial\Lambda}{\partial t}(t,T),\frac{\partial W_\Lambda}{\partial t}(t,T)\right)dt\right)\Bigg|_{T=0} = 0.$$

As before, let's carry the $\frac{d}{dT}$ derivative under the integral sign to get that:

Equation 3.3.7: $$\int_{t_0}^{t_1} \frac{\partial}{\partial T}\left(F\left(t,U(t),\Lambda(t,T),W_\Lambda(t,T),U'(t),\frac{\partial\Lambda}{\partial t}(t,T),\frac{\partial W_\Lambda}{\partial t}(t,T)\right)\right)\Bigg|_{(t,0)} dt$$

$$= \int_{t_0}^{t_1}\left(\frac{\partial F}{\partial v}\cdot\frac{\partial\Lambda}{\partial T}(t,0) + \frac{\partial F}{\partial w}\cdot\frac{\partial W_\Lambda}{\partial T}(t,0) + \frac{\partial F}{\partial v'}\cdot\frac{\partial^2\Lambda}{\partial t\partial T}(t,0) + \frac{\partial F}{\partial w'}\cdot\frac{\partial^2 W_\Lambda}{\partial t\partial T}(t,0)\right)dt = 0.$$

In the second integral above, I omitted the arguments of the partials of F in order to make the equation shorter. These partials of F here are being evaluated at:

$$\left(t, U(t), \Lambda(t,0), W_\Lambda(t,0), U'(t), \frac{\partial \Lambda}{\partial t}(t,0), \frac{\partial W_\Lambda}{\partial t}(t,0)\right).$$

By Equations 3.3.3 and 3.3.4, the get that above arguments of the partials of F are the same things as:

$$\big(t, U(t), V(t), W(t), U'(t), V'(t), W'(t)\big).$$

Now let's go back and analyze Λ and W_Λ a little bit. By Equations 3.3.3 and 3.3.4, we see that:

$$\forall t \notin [\alpha, \beta], \qquad \frac{\partial \Lambda}{\partial T}(t,0) = \frac{\partial W_\Lambda}{\partial T}(t,0) = \frac{\partial^2 \Lambda}{\partial t \partial T}(t,0) = \frac{\partial^2 W_\Lambda}{\partial t \partial T}(t,0) = 0$$

All of these terms are zero on $t \notin [\alpha, \beta]$! Now, notice that every term in the integrand of the second integral in Equation 3.3.7 is multiplied by at least one of the above four terms. So we get that on $t \notin [\alpha, \beta]$ the integrand in that integral is equal to zero. So we can reduce the domain of integration of the second integral in Equation 3.3.7 to $[\alpha, \beta]$ without changing the value of the integral:

$$\int_\alpha^\beta \left(\frac{\partial F}{\partial v} \cdot \frac{\partial \Lambda}{\partial T}(t,0) + \frac{\partial F}{\partial w} \cdot \frac{\partial W_\Lambda}{\partial T}(t,0) + \frac{\partial F}{\partial v'} \cdot \frac{\partial^2 \Lambda}{\partial t \partial T}(t,0) + \frac{\partial F}{\partial w'} \cdot \frac{\partial^2 W_\Lambda}{\partial t \partial T}(t,0)\right) dt = 0.$$

Great, let's plug in the definition of W_Λ over the interval $[\alpha, \beta]$ into the above integrand (see Equation 3.3.4). We will then get that:

$$\int_\alpha^\beta \left(\frac{\partial F}{\partial v} \cdot \frac{\partial \Lambda}{\partial T}(t,0) + \frac{\partial F}{\partial w} \cdot \frac{\partial}{\partial T}\Big(f\big(U(t), \Lambda(t,T)\big)\Big)\Big|_{(t,0)} + \frac{\partial F}{\partial v'} \cdot \frac{\partial^2 \Lambda}{\partial t \partial T}(t,T) + \frac{\partial F}{\partial w'} \cdot \frac{\partial}{\partial t \partial T}\Big(f\big(U(t), \Lambda(t,T)\big)\Big)\Big|_{(t,0)}\right) dt = 0.$$

Calculating out the $\frac{\partial}{\partial T}$ and $\frac{\partial^2}{\partial t \partial T}$ partials in the above integral gives us that:

$$\int_\alpha^\beta \left(\frac{\partial F}{\partial v} \cdot \frac{\partial \Lambda}{\partial T} + \frac{\partial F}{\partial w} \cdot \frac{\partial f}{\partial y} \cdot \frac{\partial \Lambda}{\partial T} + \frac{\partial F}{\partial v'} \cdot \frac{\partial^2 \Lambda}{\partial t \partial T} + \frac{\partial F}{\partial w'} \cdot \left(\frac{\partial^2 f}{\partial x \partial y} \cdot \frac{\partial \Lambda}{\partial T} \cdot U'(t) + \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial \Lambda}{\partial T} \cdot \frac{\partial \Lambda}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial^2 \Lambda}{\partial t \partial T}\right)\right) dt$$

$$= \int_\alpha^\beta \left(\frac{\partial F}{\partial v} \cdot \frac{\partial \Lambda}{\partial T} + \frac{\partial F}{\partial w} \cdot \frac{\partial f}{\partial y} \cdot \frac{\partial \Lambda}{\partial T} + \frac{\partial F}{\partial v'} \cdot \frac{\partial^2 \Lambda}{\partial t \partial T} + \frac{\partial F}{\partial w'} \cdot \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{\partial \Lambda}{\partial T} \cdot U'(t) + \frac{\partial F}{\partial w'} \cdot \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial \Lambda}{\partial T} \cdot \frac{\partial \Lambda}{\partial t} + \frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y} \cdot \frac{\partial^2 \Lambda}{\partial t \partial T}\right) dt = 0.$$

Here I omitted writing the arguments of the partials of f and Λ. They are being evaluated at $\big(U(t), \Lambda(t,0)\big) = \big(U(t), V(t)\big)$ and $(t, 0)$ respectively. Instead of dragging our Λ any further, let's plug in our formula for Λ in Equation 3.3.3 into the above equation. Notice that since here we are only working over the interval $[\alpha, \beta]$, we have here by Equation 3.3.3 that $\Lambda(t, T) = V(t) + \tilde{h}(t)T$ and so the above equation becomes:

$$Equation\ 3.3.8\text{:}\quad \int_{\alpha}^{\beta} \left(\frac{\partial F}{\partial v} \cdot \tilde{h}(t) + \frac{\partial F}{\partial w} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}(t) + \frac{\partial F}{\partial v'} \cdot \tilde{h}'(t) + \frac{\partial F}{\partial w'} \cdot \frac{\partial^2 f}{\partial x \partial y} \cdot \tilde{h}(t) \cdot U'(t) + \frac{\partial F}{\partial w'} \cdot \frac{\partial^2 f}{\partial y^2} \cdot \tilde{h}(t) \cdot V'(t) + \frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}'(t) \right) dt = 0.$$

Now, let's get rid of the annoying $\tilde{h}'(t)$ in the above integral. Let's do this by integrating the terms $\int_{\alpha}^{\beta} \frac{\partial F}{\partial v'} \cdot \tilde{h}'(t) dt$ and $\int_{\alpha}^{\beta} \frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}'(t) dt$ in the above integral by parts. Integrating the first of these terms by parts gives:

$$\int_{\alpha}^{\beta} \frac{\partial F}{\partial v'} \cdot \tilde{h}'(t) dt = \frac{\partial F}{\partial v'} \cdot \tilde{h}(t) \bigg|_{t=\alpha}^{t=\beta} - \int_{\alpha}^{\beta} \frac{d}{dt}\left(\frac{\partial F}{\partial v'}\right) \cdot \tilde{h}(t) dt.$$

Going back and looking at the boundary conditions that we required $\tilde{h}(t)$ to satisfy: $\tilde{h}(\alpha) = \tilde{h}(\beta) = 0$, we see that the term $\frac{\partial F}{\partial v'} \cdot \tilde{h}(t) \Big|_{t=\alpha}^{t=\beta} = 0$. So, the above equation becomes:

$$\int_{\alpha}^{\beta} \frac{\partial F}{\partial v'} \cdot \tilde{h}'(t) dt = -\int_{\alpha}^{\beta} \frac{d}{dt}\left(\frac{\partial F}{\partial v'}\right) \cdot \tilde{h}(t) dt.$$

Plugging this into Equation 3.3.8 gives us that our giant integral becomes:

$$Equation\ 3.3.9\text{:}\quad \int_{\alpha}^{\beta} \left(\frac{\partial F}{\partial v} \cdot \tilde{h}(t) + \frac{\partial F}{\partial w} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}(t) - \frac{d}{dt}\left(\frac{\partial F}{\partial v'}\right) \cdot \tilde{h}(t) + \frac{\partial F}{\partial w'} \cdot \frac{\partial^2 f}{\partial x \partial y} \cdot \tilde{h}(t) \cdot U'(t) + \frac{\partial F}{\partial w'} \cdot \frac{\partial^2 f}{\partial y^2} \cdot \tilde{h}(t) \cdot V'(t) + \frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}'(t) \right) dt = 0.$$

Now let us integrate the other term: $\int_{\alpha}^{\beta} \frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}'(t) dt$ in the above integral by parts. We have that:

$$\int_{\alpha}^{\beta} \frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}'(t) dt = \frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}(t) \bigg|_{t=\alpha}^{t=\beta} - \int_{\alpha}^{\beta} \frac{d}{dt}\left(\frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y}\right) \cdot \tilde{h}(t) dt.$$

For the same reason as above, we get that $\frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}(t) \Big|_{t=\alpha}^{t=\beta} = 0$ and so we have that:

$$\int_{\alpha}^{\beta} \frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}'(t)dt = -\int_{\alpha}^{\beta} \frac{d}{dt}\left(\frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y}\right) \cdot \tilde{h}(t)dt.$$

Applying the product rule on the $\frac{d}{dt}$ derivative gives us that:

$$\int_{\alpha}^{\beta} \frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}'(t)dt = -\int_{\alpha}^{\beta} \left(\frac{d}{dt}\left(\frac{\partial F}{\partial w'}\right) \cdot \frac{\partial f}{\partial y} + \frac{\partial F}{\partial w'} \cdot \frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\right) \cdot \tilde{h}(t)dt.$$

The derivative:

$$\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) = \frac{d}{dt}\left(\frac{\partial f}{\partial y}\big(U(t), V(t)\big)\right) = \frac{\partial^2 f}{\partial y \partial x} \cdot U'(t) + \frac{\partial^2 f}{\partial y^2} \cdot V'(t).$$

Again, the partials of f here are being evaluated at $\big(U(t), V(t)\big)$. I wrote out the arguments of f in the first equality above so as to make the differentiation easier. Plugging this into the previous equation gives us that:

$$\int_{\alpha}^{\beta} \frac{\partial F}{\partial w'} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}'(t)dt = -\int_{\alpha}^{\beta} \left(\frac{d}{dt}\left(\frac{\partial F}{\partial w'}\right) \cdot \frac{\partial f}{\partial y} + \frac{\partial F}{\partial w'} \cdot \left(\frac{\partial^2 f}{\partial y \partial x} \cdot U'(t) + \frac{\partial^2 f}{\partial y^2} \cdot V'(t)\right)\right) \cdot \tilde{h}(t)dt.$$

Plugging this into Equation 3.3.9 finally gives us that:

$$\int_{\alpha}^{\beta} \Bigg(\frac{\partial F}{\partial v} \cdot \tilde{h}(t) + \frac{\partial F}{\partial w} \cdot \frac{\partial f}{\partial y} \cdot \tilde{h}(t) - \frac{d}{dt}\left(\frac{\partial F}{\partial v'}\right) \cdot \tilde{h}(t) + \frac{\partial F}{\partial w'} \cdot \frac{\partial^2 f}{\partial x \partial y} \cdot \tilde{h}(t) \cdot U'(t) + \frac{\partial F}{\partial w'} \cdot \frac{\partial^2 f}{\partial y^2} \cdot \tilde{h}(t)$$

$$\cdot V'(t) - \left(\frac{d}{dt}\left(\frac{\partial F}{\partial w'}\right) \cdot \frac{\partial f}{\partial y} + \frac{\partial F}{\partial w'} \cdot \left(\frac{\partial^2 f}{\partial y \partial x} \cdot U'(t) + \frac{\partial^2 f}{\partial y^2} \cdot V'(t)\right)\right) \cdot \tilde{h}(t) \Bigg)dt$$

$$= 0.$$

Many of the terms on the left-hand side cancel out to finally give:

$$\int_{\alpha}^{\beta} \left(\frac{\partial F}{\partial v} - \frac{d}{dt}\left(\frac{\partial F}{\partial v'}\right) + \left(\frac{\partial F}{\partial w} - \frac{d}{dt}\left(\frac{\partial F}{\partial w'}\right)\right) \cdot \frac{\partial f}{\partial y}\right) \cdot \tilde{h}(t)dt = 0.$$

Yes! Let's review what we've proved so far. With the above equation we proved that for $\forall \tilde{h} \in C^2[\alpha, \beta]$ that satisfies the boundary conditions:

$$\tilde{h}(\alpha) = \tilde{h}'(\alpha) = \tilde{h}''(\alpha) = 0 \quad and \quad \tilde{h}(\beta) = \tilde{h}'(\beta) = \tilde{h}''(\beta) = 0,$$

the above integral is equal to zero. This means that we can apply Lemma 1.3.1 from Chapter One to finally get that:

$$\frac{\partial F}{\partial v} - \frac{d}{dt}\left(\frac{\partial F}{\partial v'}\right) + \left(\frac{\partial F}{\partial w} - \frac{d}{dt}\left(\frac{\partial F}{\partial w'}\right)\right) \cdot \frac{\partial f}{\partial y} \equiv 0$$

on all of $[\alpha, \beta]$. Fantastic! Let's rewrite this equation in more exciting notation: the variational derivative notation. Using the notation defined Definition 1.3.6 in Chapter One, we can write down (remember, all of the partials of F here are being evaluated at $\big(t, U(t), V(t), W(t), U'(t), V'(t), W'(t)\big)$):

$$\frac{\delta J}{\delta v}[U(t), V(t), W(t)] = \frac{\partial F}{\partial v} - \frac{d}{dt}\left(\frac{\partial F}{\partial v'}\right),$$
$$\frac{\delta J}{\delta w}[U(t), V(t), W(t)] = \frac{\partial F}{\partial w} - \frac{d}{dt}\left(\frac{\partial F}{\partial w'}\right).$$

So the previous equation can be rewritten as:

$$\frac{\delta J}{\delta v}[U(t), V(t), W(t)] + \frac{\delta J}{\delta w}[U(t), V(t), W(t)] \cdot \frac{\partial f}{\partial y}\big(U(t), V(t)\big) \equiv 0$$

on $[\alpha, \beta]$. I decided to include the arguments of $\partial f / \partial y$ in this equation. Of more interest to us is that since $t_2 \in [\alpha, \beta]$ (because the interval $[\alpha, \beta]$ was centered at t_2), we get that the above equation holds at $t = t_2$:

$$\frac{\delta J}{\delta v}[U(t_2), V(t_2), W(t_2)] + \frac{\delta J}{\delta w}[U(t_2), V(t_2), W(t_2)] \cdot \frac{\partial f}{\partial y}\big(U(t_2), V(t_2)\big) = 0.$$

Wow! With this equation we've been able to relate how the "variational partials" of J (the components of $\nabla_\delta J$ that is) relate to the partials of f. From here it is in fact possible to obtain yet another equation for free using the symmetry of the variables x and y. Notice that if we would go through the above proof again but would switch the roles of x and y, u and v, and U and V, we would instead arrive at the equation:

$$\frac{\delta J}{\delta u}[U(t_2), V(t_2), W(t_2)] + \frac{\delta J}{\delta w}[U(t_2), V(t_2), W(t_2)] \cdot \frac{\partial f}{\partial x}\big(U(t_2), V(t_2)\big) = 0.$$

The only difference here is that $\frac{\delta J}{\delta v}$ is now $\frac{\delta J}{\delta u}$ and $\frac{\partial f}{\partial y}$ is now $\frac{\partial f}{\partial x}$. So, we have that the following system of equations holds at $t = t_2$:

$$\frac{\delta J}{\delta u}[U(t_2), V(t_2), W(t_2)] + \frac{\delta J}{\delta w}[U(t_2), V(t_2), W(t_2)] \cdot \frac{\partial f}{\partial x}\big(U(t_2), V(t_2)\big) = 0,$$
$$\frac{\delta J}{\delta v}[U(t_2), V(t_2), W(t_2)] + \frac{\delta J}{\delta w}[U(t_2), V(t_2), W(t_2)] \cdot \frac{\partial f}{\partial y}\big(U(t_2), V(t_2)\big) = 0.$$

Great! From here in order to finish off the proof of this theorem we now need to show how the above equations relate to the partials of g. By the implicit function theorem (while remembering that $W(t_2) = f\big(U(t_2), V(t_2)\big)$ since $p = \big(U(t_2), V(t_2), W(t_2)\big)$ lies on the surface S) we know that:

$$\frac{\partial f}{\partial x}\big(U(t_2), V(t_2)\big) = -\frac{\frac{\partial g}{\partial x}\big(U(t_2), V(t_2), W(t_2)\big)}{\frac{\partial g}{\partial z}\big(U(t_2), V(t_2), W(t_2)\big)},$$

$$\frac{\partial f}{\partial y}\big(U(t_2), V(t_2)\big) = -\frac{\frac{\partial g}{\partial y}\big(U(t_2), V(t_2), W(t_2)\big)}{\frac{\partial g}{\partial z}\big(U(t_2), V(t_2), W(t_2)\big)}.$$

Plugging this into the previous two equations gives that:

$$\frac{\delta J}{\delta u}[U(t_2), V(t_2), W(t_2)] - \frac{\delta J}{\delta w}[U(t_2), V(t_2), W(t_2)] \cdot \frac{\frac{\partial g}{\partial x}\big(U(t_2), V(t_2), W(t_2)\big)}{\frac{\partial g}{\partial z}\big(U(t_2), V(t_2), W(t_2)\big)} = 0,$$

$$\frac{\delta J}{\delta v}[U(t_2), V(t_2), W(t_2)] - \frac{\delta J}{\delta w}[U(t_2), V(t_2), W(t_2)] \cdot \frac{\frac{\partial g}{\partial y}\big(U(t_2), V(t_2), W(t_2)\big)}{\frac{\partial g}{\partial z}\big(U(t_2), V(t_2), W(t_2)\big)} = 0.$$

Or after rearrangement:

$$\frac{\delta J}{\delta u}[U(t_2), V(t_2), W(t_2)] = \frac{\frac{\delta J}{\delta w}[U(t_2), V(t_2), W(t_2)]}{\frac{\partial g}{\partial z}\big(U(t_2), V(t_2), W(t_2)\big)} \cdot \frac{\partial g}{\partial x}\big(U(t_2), V(t_2), W(t_2)\big),$$

$$\frac{\delta J}{\delta v}[U(t_2), V(t_2), W(t_2)] = \frac{\frac{\delta J}{\delta w}[U(t_2), V(t_2), W(t_2)]}{\frac{\partial g}{\partial z}\big(U(t_2), V(t_2), W(t_2)\big)} \cdot \frac{\partial g}{\partial y}\big(U(t_2), V(t_2), W(t_2)\big).$$

Take the function $\lambda(t)$ from the statement of the theorem and at $t = t_2$ define it to be:

$$\lambda(t_2) = \frac{\frac{\delta J}{\delta w}[U(t_2), V(t_2), W(t_2)]}{\frac{\partial g}{\partial z}\big(U(t_2), V(t_2), W(t_2)\big)}$$

Then notice that the previous equation implies that the following three equations hold:

$$\frac{\delta J}{\delta u}[U(t_2), V(t_2), W(t_2)] = \lambda(t_2) \cdot \frac{\partial g}{\partial x}\big(U(t_2), V(t_2), W(t_2)\big),$$

$$\frac{\delta J}{\delta v}[U(t_2), V(t_2), W(t_2)] = \lambda(t_2) \cdot \frac{\partial g}{\partial y}\big(U(t_2), V(t_2), W(t_2)\big),$$

$$\frac{\delta J}{\delta w}[U(t_2), V(t_2), W(t_2)] = \lambda(t_2) \cdot \frac{\partial g}{\partial z}\big(U(t_2), V(t_2), W(t_2)\big).$$

Repeat the above proof for every $t_2 \in (t_0, t_1)$ (remember that $t_2 \in (t_0, t_1)$ was chosen arbitrarily in the beginning of this proof) to get that the following system of equations holds:

$$\frac{\delta J}{\delta u}[U(t), V(t), W(t)] = \lambda(t) \cdot \frac{\partial g}{\partial x}\big(U(t), V(t), W(t)\big),$$
$$\frac{\delta J}{\delta v}[U(t), V(t), W(t)] = \lambda(t) \cdot \frac{\partial g}{\partial y}\big(U(t), V(t), W(t)\big),$$
$$\frac{\delta J}{\delta w}[U(t), V(t), W(t)] = \lambda(t) \cdot \frac{\partial g}{\partial z}\big(U(t), V(t), W(t)\big).$$

on all of $t \in (t_0, t_1)$ for some function $\lambda(t)$. Notice that this system of equations can be written down in the vector form:

$$\left(\frac{\delta J}{\delta u}[U(t), V(t), W(t)], \frac{\delta J}{\delta v}[U(t), V(t), W(t)], \frac{\delta J}{\delta w}[U(t), V(t), W(t)]\right)$$
$$= \lambda(t)\left(\frac{\partial g}{\partial x}\big(U(t), V(t), W(t)\big), \frac{\partial g}{\partial y}\big(U(t), V(t), W(t)\big), \frac{\partial g}{\partial z}\big(U(t), V(t), W(t)\big)\right).$$

on all of $t \in (t_0, t_1)$. With this we have shown that our extremum curve satisfies the Euler-Lagrange vector differential equation, which is what we wanted to prove! Since the left-hand side of the above equation is the variational gradient of J and $\lambda(t)$ is being multiplied by the gradient of g on the right-hand side, the above equation can be rewritten in the beautiful form:

$$\nabla_\delta J[U, V, W] = \lambda(t) \nabla g\big(U(t), V(t), W(t)\big).$$

If we're too lazy to write out the arguments of $\nabla_\delta J$ and ∇g, then the above equation takes the nice to look at form:

$$\nabla_\delta J = \lambda(t) \nabla g.$$

With this we have proven the theorem.

■

We will do an example of an application of the above theorem when we will prove the minimizing curve theorem in Chapter 5. The above theorem will provide us with the most elegant proof of that theorem.

As discussed before, a surface S that is the graph of a differentiable function $z = f(x, y)$ can always be represented in the form that's stated in the above theorem. Indeed, to see this just set:

$$g(x, y, z) = z - f(x, y)$$

and notice that now S is now the set of points (x, y, z) such that $g(x, y, z) = z - f(x, y) = 0$. More formally, we have that:

$$S = \{(x, y, z) : g(x, y, z) = 0\}$$

So if J is a functional of the form as in the statement of the above theorem in relation to this S, then the vector differential equation in the conclusion of that theorem takes the form:

$$\nabla_\delta J = \lambda(t)\nabla g = \lambda(t)\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

The above theorem has an analog in higher dimensions as well. Notice that in the statement and proof of the above theorem, not much was used in the way of the fact that we were working with three-dimensional surfaces. As a result, the above theorem and proof can be extended into higher dimensions as follows.

Theorem 3.3.10 (Hypersurface Euler-Lagrange Vector Differential Equation): *Suppose that we have a hypersurface*[20] *S that is the level set of some continuously differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ at zero. In other words:*

$$S = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : g(x_1, x_2, \dots, x_n) = c\}.$$

where c is some fixed constant. Suppose also that ∇g never vanishes on S. Let $\left(A_{x_1}, A_{x_2}, \dots, A_{x_n}\right)$ and $\left(B_{x_1}, B_{x_2}, \dots, B_{x_n}\right)$ are two points on the surface S. Now, let J be a functional of the form:

$$J[u_1(t), u_2(t), \dots, u_n(t)] = \int_{t_0}^{t_1} F\left(t, u_1(t), u_2(t), \dots, u_n(t), u_1'(t), u_2'(t), \dots, u_n'(t)\right)dt$$

where $F \in C^2[\mathbb{R}^{2n+1}]$ and where J's domain is the set of curves $(u_1, u_2, \dots, u_n) \in \prod_{k=1}^{n} C^2[t_0, t_1]$ that satisfy the boundary conditions:

$$\left(u_1(t_0), u_2(t_0), \dots, u_n(t_0)\right) = \left(A_{x_1}, A_{x_2}, \dots, A_{x_n}\right),$$

$$\left(u_1(t_1), u_2(t_1), \dots, u_n(t_1)\right) = \left(B_{x_1}, B_{x_2}, \dots, B_{x_n}\right),$$

and that lie on the surface S:

$$\forall t \in [t_0, t_1], \qquad g\left(u_1(t), u_2(t), \dots, u_n(t)\right) = c.$$

Now suppose that the curve $\left(U_1(t), U_2(t), \dots, U_n(t)\right)$ is a local extremum of J. Then this local extremum curve must satisfy the equation:

$$\nabla_\delta J[U_1, U_2, \dots, U_n] = \lambda(t)\nabla g\left(U_1(t), U_2(t), \dots, U_n(t)\right)$$

for some real valued function $\lambda(t)$. If we don't want to write out some of the arguments in the above equation, this equation can be rewritten in the nicer to look at form:

$$\nabla_\delta J = \lambda(t)\nabla g,$$

[20] "hypersurface" basically just means a surface but in higher dimensions.

The nice thing about this form of the equation is that this equation looks exactly like the equation in the conclusion of the previous theorem.

Proof: I will leave the proof of this theorem to the reader as an exercise. This theorem is proved just like the previous theorem except that here you will have to do things in higher dimensions and with more variables involved. Borrowing notation from the poof of the previous theorem, you're going to want to construct a compact box R such that S locally to p is the graph of a function $f(x_1, x_2, \dots, x_{n-1})$. Then you'll have to construct a flow $\Lambda(t, T)$ that passes through $U_{n-1}(t)$ at flow time $T = 0$ and that constantly stay in the smaller box R_s. After that you will need to consider the surface flow:

$$\big(U_1(t), U_2(t), \dots, U_{n-2}(t), \Lambda(t, T), W_\Lambda(t, T)\big)$$

where:

$$W_\Lambda(t, T) = \begin{cases} f\big(U_1(t), U_2(t), \dots, U_{n-2}(t), \Lambda(t, T)\big) & if \quad t \in [\alpha, \beta] \\ W(t) & if \quad t \notin [\alpha, \beta] \end{cases}$$

Then continue from there like we did in the proof of the previous theorem. A main difficulty in the proof of this theorem will be to keep track of all of the variables involved.

■

Section 4: Cool Applications of Variational Theory: Green's Theorem and the Divergence Theorem

Some time ago, I was sitting in my second-year calculus course and my teacher was teaching us Green's Theorem. Before he showed us a proof of this theorem he mentioned that Cauchy in his 1814 memoir provided several arguments as proofs of this theorem. The reason why he included several arguments, my teacher said, was because he wasn't entirely sure whether each argument was 100% correct and so multiple approaches that indicated the same answer made him more convinced of the fact that this theorem is true. My teacher said that one of Cauchy "proofs" involved approximating the region of integration with boxes. Another argument that Cauchy provided, my teacher said, used the calculus of variations. The moment my teacher said the words "calculus of variations," I got really excited about the possibility of proving Green's Theorem through a variational approach. About a year later I thought of a variational proof of Green's theorem and I would like to present it to you.

Green's Theorem states that for any suitable region Ω ("suitable" is hard to describe in this situation, but we'll come back to this) and any $P, Q \in C^1[\Omega]$,

$$\iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_{\partial\Omega} (P, Q) \cdot (dx, dy).$$

Both quantities of the above equation can be interpreted as functionals over a space of closed looped curves. Indeed, let E be the set of all suitable non-self-intersecting closed looped parametrized curves. For any $\gamma \in E$, let $\gamma^{[enc]}$ denote the region enclosed within the curve γ. Now, let J and K be two functionals over E defined by:

$$\forall \gamma \in E, \qquad J[\gamma] = \iint\limits_{\gamma^{[enc]}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy,$$

$$\forall \gamma \in E, \qquad K[\gamma] = \oint\limits_{\gamma} (P, Q) \cdot (dx, dy).$$

We can then restate Green's Theorem into the statement that these two functionals are equal everywhere. More explicitly: for any $\gamma \in E$,

$$J[\gamma] = K[\gamma].$$

Thus, we can reformulate the statement of Green's theorem into a statement about a functional equality.

How does one in general prove that two functionals are equal everywhere? Well, there are many ways. To find these ways, let's look at how we do a similar sort of thing with multivariable functions. How does one for example prove that two differentiable functions $F, G : \mathbb{R}^2 \to \mathbb{R}$ are equal everywhere? One way is to show that their gradients are equal on all of $\mathbb{R}^2$ and then proceed to find one point where the two functions are equal. This will imply that F and G are equal on all of $\mathbb{R}^2$. The same principle in fact works with functionals. Up to technical refinement, if you show that the variational derivatives of two functionals are equal everywhere and the two functionals agree on some curve, then the two functionals are in fact equal everywhere. We will come back to this idea in another section.

However, there is another very closely related way that one can use in order to show that two functionals are equal. In the case of the multivariable functions F and G, a way to show that these two functions are equal goes as follows. First find a point (x_0, y_0) where they are equal: $F(x_0, y_0) = G(x_0, y_0)$. Then, in order to show that F and G are equal at any point $(x, y) \in \mathbb{R}^n$ consider the line $l(t)$ that goes from the point (x_0, y_0) to the point (x, y). If the following holds

$$\frac{d}{dt}\left[F\big(l(t)\big)\right] = \frac{d}{dt}\left[G\big(l(t)\big)\right]$$

constantly as $l(t)$ travels from (x_0, y_0) to (x, y), then $F(x, y) = G(x, y)$. Why? This is just a direct application of the Fundamental Theorem of Calculus (let t_0 and t_1 here denote the times when $l(t)$ crosses (x_0, y_0) and (x, y) respectively: $l(t_0) = (x_0, y_0)$, $l(t_1) = (x, y)$):

$$F(x, y) = F\big(l(t_1)\big) = \left[F\big(l(t_1)\big) - F\big(l(t_0)\big)\right] + F(x_0, y_0) = \int\limits_{t_0}^{t_1} \frac{d}{dt}\left[F\big(l(t)\big)\right]dt + F(x_0, y_0) =$$

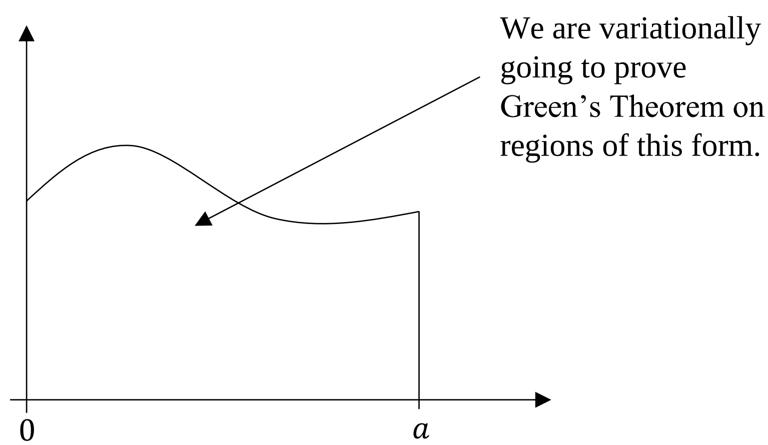
$$\int\limits_{t_0}^{t_1} \frac{d}{dt}\left[G\big(l(t)\big)\right]dt + G(x_0, y_0) = \left[G\big(l(t_1)\big) - G\big(l(t_0)\big)\right] + G(x_0, y_0) = G(x, y).$$

And so:

$$F(x,y) = G(x,y).$$

The exact same thing can be done with functionals except in this case we are going to have to use the lines in the space of curves, which are the linear flows. This is the approach that we will take in our variational proof of Green's Theorem.

Let's prove a little bit simpler version of Green's Theorem. Since the set of all regions that Green's Theorem applies to is very hard to describe, we have to resort to some simplification of the regions that we consider in our proof of Green's Theorem. So let's prove Green's Theorem on regions that lie between the x-axis and the graph of a non-negative continuously differentiable function.



After that I will show you an argument that allows you to extend our result of Green's Theorem to certain more complicated regions that can't necessarily be represented in this form.

Theorem 3.4.1 (Graph Region Version of Green's Theorem): *Suppose that we have a non-negative continuously differentiable function $g(x) \geq 0$ over a compact interval $[0,a]$ and let Ω be the region between the x-axis and $g(x)$:*

$$\Omega = \{(x,y) : x \in [0,a], 0 \leq y \leq g(x)\}$$

(see the image above). Then,

$$\iint\limits_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint\limits_{\partial\Omega} (P,Q) \cdot (dx,dy).$$

Proof: We will prove this through functional equalities. For any function/curve $h(x)$ over $[0,a]$, let Ω_h denote the region between the x-axis and $g(x)$:

$$\Omega_h = \{(x,y) : x \in [0,a], 0 \leq y \leq h(x)\}.$$

Notice that with our notation, $\Omega_g = \Omega$. Now, let us define the functionals (here I use $h(x)$ for the variable for our two functionals):

$$J[h] = \iint_{\Omega_h} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$K[h] = \oint_{\partial\Omega_h} (P, Q) \cdot (dx, dy)$$

where the domain of J and K is the set of all non-negative continuously differentiable functions/curves $h : [0, a] \to \mathbb{R}$. Let's write out the above two integrals in a little bit more explicit forms. We have that:

$$J[h] = \iint_{\Omega_h} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_0^a \int_0^{h(x)} \left(\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y)\right) dydx,$$

and (here the convention is to integrate contour integrals counterclockwise):

$$K[h] = \oint_{\partial\Omega_h} (P, Q) \cdot (dx, dy)$$

$$= \int_0^a P(x, 0)dx + \int_0^{h(a)} Q(a, y)dy - \int_0^a \Big(P\big(x, h(x)\big) + Q\big(x, h(x)\big)h'(x)\Big)\, dx - \int_0^{h(0)} Q(0, y)dy.$$

Great! Notice that J and K are trivially equal at the curve $h \equiv 0$ since in this case Ω_h is just the rectangle with height 0 and width a. In other words:

$$J[h \equiv 0] = K[h \equiv 0]$$

(both sides are zero in fact). So we have found our curve of agreement. Now all that's left to do is to take the linear flow Λ that goes between the curves $h \equiv 0$ and $g(x)$ and show that the time derivatives of $J[\Lambda(x, t)]$ and $K[\Lambda(x, t)]$ are constantly equal along this linear flow. Let's do this! Take the linear flow $\Lambda : [0, a] \times [0, 1] \to \mathbb{R}$ defined by:

$$\Lambda(x, t) = g(x)t.$$

Notice that this linear flow does go between the curves $h \equiv 0$ and $g(x)$ because $\Lambda(x, 0) \equiv 0$ and $\Lambda(x, 1) = g(x)$. Now we get to the fun part! We need to show that:

$Equation$ 3.4.2: $$\frac{d}{dt}(J[\Lambda(x, t)]) = \frac{d}{dt}(K[\Lambda(x, t)])$$

constantly on $t \in (0, 1)$. Let's first compute the left-hand side of the above equation. We have that (in these calculations I will use Leibniz's rule for differentiating integrals):

$$\frac{d}{dt}(J[\Lambda(x, t)]) = \frac{d}{dt}\left(\iint_{\Omega_{\Lambda(x,t)}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy\right)$$

$$= \frac{d}{dt}\left(\int_0^a \int_0^{\Lambda(x,t)} \left(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right) dy dx\right)$$

$$= \int_0^a \frac{d}{dt}\left(\int_0^{\Lambda(x,t)} \left(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right) dy\right) dx$$

$$= \int_0^a \left(\frac{\partial Q}{\partial x}\big(x,\Lambda(x,t)\big) - \frac{\partial P}{\partial y}\big(x,\Lambda(x,t)\big)\right)\frac{\partial \Lambda(x,t)}{\partial t} dx.$$

And so, we get that:

$$\textit{Equation } 3.4.3: \quad \frac{d}{dt}\big(J[\Lambda(x,t)]\big) = \int_0^a \left(\frac{\partial Q}{\partial x}\big(x,\Lambda(x,t)\big) - \frac{\partial P}{\partial y}\big(x,\Lambda(x,t)\big)\right)\frac{\partial \Lambda(x,t)}{\partial t} dx.$$

Now let's compute the right-hand side of Equation 3.4.2. We have that:

$$\frac{d}{dt}\big(K[\Lambda(x,t)]\big) = \frac{d}{dt}\left(\oint_{\partial\Omega_{\Lambda(x,t)}} (P,Q)\cdot(dx,dy)\right)$$

$$= \frac{d}{dt}\left(\int_0^a P(x,0)dx + \int_0^{\Lambda(a,t)} Q(a,y)dy - \int_0^a \left(P\big(x,\Lambda(x,t)\big) + Q\big(x,\Lambda(x,t)\big)\frac{\partial \Lambda(x,t)}{\partial x}\right) dx \right.$$
$$\left. - \int_0^{\Lambda(0,t)} Q(0,y)dy\right)$$

$$= 0 + Q\big(a,\Lambda(a,t)\big)\frac{\partial \Lambda(a,t)}{\partial t} - \int_0^a \frac{d}{dt}\left(P\big(x,\Lambda(x,t)\big) + Q\big(x,\Lambda(x,t)\big)\frac{\partial \Lambda(x,t)}{\partial x}\right) dx$$
$$- Q\big(0,\Lambda(0,t)\big)\frac{\partial \Lambda(0,t)}{\partial t}$$

$$= Q\big(a,\Lambda(a,t)\big)\frac{\partial \Lambda(a,t)}{\partial t} - Q\big(0,\Lambda(0,t)\big)\frac{\partial \Lambda(0,t)}{\partial t}$$
$$- \int_0^a \left(\frac{\partial P}{\partial y}\big(x,\Lambda(x,t)\big)\frac{\partial \Lambda(x,t)}{\partial t} + \frac{\partial Q}{\partial y}\big(x,\Lambda(x,t)\big)\frac{\partial \Lambda(x,t)}{\partial x}\frac{\partial \Lambda(x,t)}{\partial t} + Q\big(x,\Lambda(x,t)\big)\frac{\partial^2 \Lambda(x,t)}{\partial x \partial t}\right) dx$$

$$= Q\big(a,\Lambda(a,t)\big)\frac{\partial \Lambda(a,t)}{\partial t} - Q\big(0,\Lambda(0,t)\big)\frac{\partial \Lambda(0,t)}{\partial t} - \int_0^a \frac{\partial P}{\partial y}\big(x,\Lambda(x,t)\big)\frac{\partial \Lambda(x,t)}{\partial t} dx$$

$$-\int_0^a \frac{\partial Q}{\partial y}\big(x,\Lambda(x,t)\big)\frac{\partial\Lambda(x,t)}{\partial x}\frac{\partial\Lambda(x,t)}{\partial t}dx - \int_0^a Q\big(x,\Lambda(x,t)\big)\frac{\partial^2\Lambda(x,t)}{\partial x\partial t}dx.$$

And so we get that:

Equation 3.4.4:
$$\frac{d}{dt}(K[\Lambda(x,t)]) = Q\big(a,\Lambda(a,t)\big)\frac{\partial\Lambda(a,t)}{\partial t} - Q\big(0,\Lambda(0,t)\big)\frac{\partial\Lambda(0,t)}{\partial t}$$
$$-\int_0^a \frac{\partial P}{\partial y}\big(x,\Lambda(x,t)\big)\frac{\partial\Lambda(x,t)}{\partial t}dx - \int_0^a \frac{\partial Q}{\partial y}\big(x,\Lambda(x,t)\big)\frac{\partial\Lambda(x,t)}{\partial x}\frac{\partial\Lambda(x,t)}{\partial t}dx$$
$$-\int_0^a Q\big(x,\Lambda(x,t)\big)\frac{\partial^2\Lambda(x,t)}{\partial x\partial t}dx.$$

In Equation 3.4.3 we were able to write out the time derivative of $J[\Lambda(x,t)]$ as an integral whose integrand only contained the $\frac{\partial\Lambda(x,t)}{\partial t}$ partial of Λ. In our attempt to show that the time derivative of $K[\Lambda(x,t)]$ is equal to the time derivative of $J[\Lambda(x,t)]$, let us try and get rid of the $\frac{\partial\Lambda(x,t)}{\partial x}$ and $\frac{\partial^2\Lambda(x,t)}{\partial x\partial t}$ partials of Λ in the above equation. Let's start doing this by looking at the $\int_0^a Q\big(x,\Lambda(x,t)\big)\frac{\partial^2\Lambda(x,t)}{\partial x\partial t}dx$ term on the right-hand side of the above equation. Integrating this term by parts gives:

$$\int_0^a Q\big(x,\Lambda(x,t)\big)\frac{\partial^2\Lambda(x,t)}{\partial x\partial t}dx = Q\big(x,\Lambda(x,t)\big)\frac{\partial\Lambda(x,t)}{\partial t}\bigg|_{x=0}^{x=a} - \int_0^a \frac{\partial}{\partial x}\Big(Q\big(x,\Lambda(x,t)\big)\Big)\frac{\partial\Lambda(x,t)}{\partial t}dx$$
$$= Q\big(a,\Lambda(a,t)\big)\frac{\partial\Lambda(a,t)}{\partial t} - Q\big(0,\Lambda(0,t)\big)\frac{\partial\Lambda(0,t)}{\partial t}$$
$$-\int_0^a \left(\frac{\partial Q}{\partial x}\big(x,\Lambda(x,t)\big) + \frac{\partial Q}{\partial y}\big(x,\Lambda(x,t)\big)\frac{\partial\Lambda(x,t)}{\partial x}\right)\frac{\partial\Lambda(x,t)}{\partial t}dx.$$

And so we get that our term is equal to:

$$\int_0^a Q\big(x,\Lambda(x,t)\big)\frac{\partial^2\Lambda(x,t)}{\partial x\partial t}dx$$
$$= Q\big(a,\Lambda(a,t)\big)\frac{\partial\Lambda(a,t)}{\partial t} - Q\big(0,\Lambda(0,t)\big)\frac{\partial\Lambda(0,t)}{\partial t}$$
$$-\int_0^a \left(\frac{\partial Q}{\partial x}\big(x,\Lambda(x,t)\big) + \frac{\partial Q}{\partial y}\big(x,\Lambda(x,t)\big)\frac{\partial\Lambda(x,t)}{\partial x}\right)\frac{\partial\Lambda(x,t)}{\partial t}dx.$$

Plugging this back into Equation 3.4.4 gives us that:

$$\frac{d}{dt}(K[\Lambda(x,t)]) = Q\big(a,\Lambda(a,t)\big)\frac{\partial\Lambda(a,t)}{\partial t} - Q\big(0,\Lambda(0,t)\big)\frac{\partial\Lambda(0,t)}{\partial t}$$

$$-\int_0^a \frac{\partial P}{\partial y}\big(x,\Lambda(x,t)\big)\frac{\partial\Lambda(x,t)}{\partial t}dx - \int_0^a \frac{\partial Q}{\partial y}\big(x,\Lambda(x,t)\big)\frac{\partial\Lambda(x,t)}{\partial x}\frac{\partial\Lambda(x,t)}{\partial t}dx$$

$$-Q\big(a,\Lambda(a,t)\big)\frac{\partial\Lambda(a,t)}{\partial t} + Q\big(0,\Lambda(0,t)\big)\frac{\partial\Lambda(0,t)}{\partial t}$$

$$+\int_0^a \left(\frac{\partial Q}{\partial x}\big(x,\Lambda(x,t)\big) + \frac{\partial Q}{\partial y}\big(x,\Lambda(x,t)\big)\frac{\partial\Lambda(x,t)}{\partial x}\right)\frac{\partial\Lambda(x,t)}{\partial t}dx.$$

Notice that a lot of the terms on the right-hand side cancel out to finally give us:

$$\frac{d}{dt}(K[\Lambda(x,t)]) = \int_0^a \left(\frac{\partial Q}{\partial x}\big(x,\Lambda(x,t)\big) - \frac{\partial P}{\partial y}\big(x,\Lambda(x,t)\big)\right)\frac{\partial\Lambda(x,t)}{\partial t}dx.$$

The same thing that we got for $\frac{d}{dt}(J[\Lambda(x,t)])$ in Equation 3.4.3! With this we have shown that the time derivatives of $J[\Lambda(x,t)]$ and $K[\Lambda(x,t)]$ are constantly equal along the linear flow Λ as $t \in (0,1)$. As with the case of the example of the multivariable functions above, this implies that $J[g] = K[g]$ since (remember $J[h \equiv 0] = K[h \equiv 0]$):

$$J[g] = J[\Lambda(x,1)] = (J[\Lambda(x,1)] - J[\Lambda(x,0)]) + J[h \equiv 0] = \int_0^1 \frac{d}{dt}(J[\Lambda(x,t)])dt + J[h \equiv 0]$$

$$= \int_0^1 \frac{d}{dt}(K[\Lambda(x,t)])dt + K[h \equiv 0] = (K[\Lambda(x,1)] - K[\Lambda(x,0)]) + K[h \equiv 0] = K[g]$$

and so:

$$J[g] = K[g].$$

Since $\Omega = \Omega_g$, we get that by plugging in the definition of J and K into the above equation finally gives us the conclusion that we want:

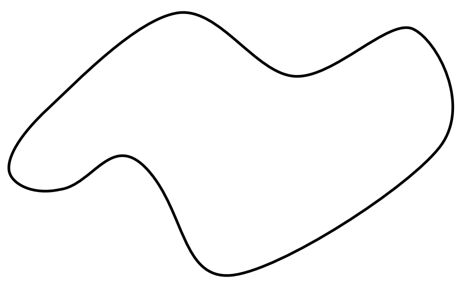
$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \oint_{\partial\Omega}(P,Q)\cdot(dx,dy)$$

With this we have proven the theorem.

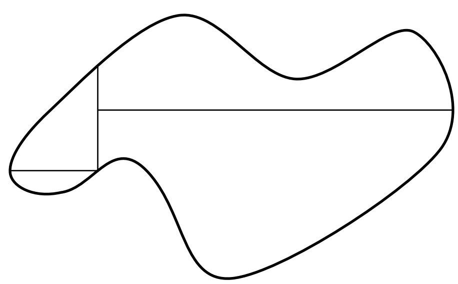
∎

A remark is in order. First of all, it is true that Green's theorem has a much easier and still completely rigorous proof over the regions described in the above theorem. This easier proof merely consists of computing the contour integral and using the Fundamental Theorem of Calculus to show that it is equal to the double integral over the region Ω. However, the primary reason that I showed you the above proof was not merely to give another proof of Green's Theorem, but instead it's there to demonstrate a technique that exists in the calculus of variations that allows you to prove many interesting equalities involving integrals. As the proof of the above theorem demonstrates, sometimes in order to prove that a certain integral equality holds it is enough to show that the equality holds on one specific curve and then show by ways of flows that they hold on the whole space of curves. This is an important concept because we will later use a similar idea to prove a version of a corollary of the Global Gauss-Bonnet Theorem.

We've proven Green's theorem on regions that, possibly after a rotation if necessary, can be represented as a region between the x-axis and the graph of a non-negative continuously differentiable function. Let's call such regions "**simple regions**" for short. How does one use this result to extend Green's Theorem to more complicated regions such as:



Notice that there is no way, even after a rotation, to represent this region as a region between the x-axis and the graph of a non-negative continuously differentiable function. The first step to showing that Green's Theorem holds on such nice regions is to first break up such a region into many smaller regions that are simple regions, like so:



Notice that each sub-region in the above picture can be represented (perhaps after a rotation) as a region as described in Theorem 3.4.1. Then, if you sum the double integrals and the contour integrals over all of the sub-regions you will get the conclusion of Green's Theorem on the whole of the above region (all of the line integrals along the interior straight lines in the above picture will cancel out since in the sum of the contour integrals you will always integrate in each direction once along each such interior line, thus causing a cancelation). Thus, we get that Green's Theorem holds on the region in the above picture.

Not all regions can be broken up in the above fashion. However, for most regions that appear in practice it is possible to break up the region just like we did with the region in the above picture. Thus, the above procedure is a good way to rigorously prove that Green's Theorem holds on many of the regions that we encounter in life.

The Divergence Theorem in $\mathbb{R}^3$ can be proved in exactly the same way as we proved Green's Theorem above for simple regions except that in this case simple regions will be regions between the x-y plane and the graph of a continuously differentiable function $h(x,y)$ (which colloquially speaking is a surface). Then you'll have to take the linear flow of "surfaces":

$$\Lambda(x,y,t) = h(x,y)t,$$

and show that the time derivative of both sides of the statement of the Divergence Theorem evaluated at the region between the x-y plane and the surface $h(x,y)t$ are always equal. Just like above this will prove the Divergence Theorem for simple regions. The extension of the Divergence Theorem to more complicated regions follows a similar argument as the one given above where you have to break up your region into many smaller sub-regions that are themselves simple regions. From there you will get the result of the Divergence Theorem on your more complicated region. And of course, there's no reason to stop at $\mathbb{R}^3$ since the above arguments work perfectly to prove the Divergence Theorem in $\mathbb{R}^n$ for general $n \in \mathbb{Z}_+$.

I will not go through the variational proof of the Divergence Theorem since in principle it's the same as the above variational proof of Green's Theorem. For this reason, I will leave the variational proof of the Divergence Theorem as an exercise to the reader.

Section 5: Cool Applications of Variational Theory: Principle of Least Action

[See future edition of this book]

Section 6: Another Approach to Proving the Equality of Two Functionals

In section 4 I indicated that there is a way to prove that two functionals are equal if their variational derivatives are equal everywhere and the two functionals are equal at some curve in their domains. Let's explore this question in detail.

To explore this approach to proving that two functionals are equal, let's again go back and see how we did this in the case of multivariable functions. How does one prove for example that two differentiable functions $F, G : \mathbb{R}^2 \to \mathbb{R}$ are equal? One way, as already mention in section 4, is to show that their gradients are equal on all of $\mathbb{R}^2$ and then proceed to find one point (x_0, y_0) where the two functions are equal. This will imply that F and G are equal on all of $\mathbb{R}^2$. Why? Well suppose that $\nabla F = \nabla G$ on all of $\mathbb{R}^2$ and that $F(x_0, y_0) = G(x_0, y_0)$. For $\forall (x, y) \in \mathbb{R}^2$ take the line:

$$l(t) = (x_0, y_0)(1-t) + (x,y)t$$

that goes from the point (x_0, y_0) to the point (x,y) as t runs from $t = 0$ to $t = 1$. From here we can see that (here I'm merely applying the Fundamental Theorem of Calculus. Also, keep in mind that $l(t)$ is a function of the form $l : [0,1] \to \mathbb{R}^2$):

$$F(x,y) = F\big(l(1)\big) = \big[F\big(l(1)\big) - F\big(l(0)\big)\big] + F(x_0,y_0) = \int_0^1 \frac{d}{dt}\big[F\big(l(t)\big)\big]dt + F(x_0,y_0)$$

$$= \int_0^1 \nabla F\big(l(t)\big) \cdot l'(t)dt + F(x_0,y_0) = \int_0^1 \nabla G\big(l(t)\big) \cdot l'(t)dt + G(x_0,y_0)$$

$$= \int_0^1 \frac{d}{dt}\big[G\big(l(t)\big)\big]dt + G(x_0,y_0) = \big[G\big(l(1)\big) - G\big(l(0)\big)\big] + G(x_0,y_0) = G\big(l(1)\big) = G(x,y).$$

And so, $F(x,y) = G(x,y)$. This means that F and G are equal on all of $\mathbb{R}^2$.

The same idea in fact works with functionals. Again, as mentioned before, up to technical refinement if you show that the variational derivatives of two functionals are equal everywhere and the two functionals agree on some curve, then the two functionals are in fact equal everywhere. Let's state and prove this rigorously in the next theorem.

In the proof of this theorem, we will need the following important lemma.

Lemma 3.6.1: *Let J be the functional:*

$$J[y] = \int_a^b F(x,y,y')dx$$

where $F, G \in C^2[\mathbb{R}^3]$. Let J's domain be the set of curves $y(x) \in C^2[a,b]$ that satisfy the boundary conditions:

$$y(a) = A \quad \text{and} \quad y(b) = B$$

where A and B are two real numbers. Suppose also that $\Lambda : [a,b] \times [t_0, t_1] \to \mathbb{R}$ is a 2-smooth flow of curves in the domain of J. Being a flow in the domain of J merely means that for every fixed flow time $t \in [t_0, t_1]$, $\Lambda(x,t)$ is a curve in the domain of J. Then,

$$\frac{d}{dt}(J[\Lambda(x,t)]) = \int_a^b \frac{\delta J}{\delta y}[\Lambda(x,t)] \cdot \frac{\partial \Lambda(x,t)}{\partial t} dt.$$

If you understood the inner product interpretation of Lemma 1.3.1 in Chapter One, then this equation can be rewritten as:

$$\frac{d}{dt}(J[\Lambda(x,t)]) = \langle \frac{\delta J}{\delta y}[\Lambda(x,t)], \frac{\partial \Lambda(x,t)}{\partial t} \rangle.$$

Proof: The proof of this lemma is merely a calculation. We in fact proved a special case of this formula in the proof of Theorem 1.3.3. In fact, the following is a mimic of the calculation that we did in the proof of Theorem 1.3.3 after the proof of Lemma 1.3.5. We have that:

$$\frac{d}{dt}(J[\Lambda(x,t)]) = \frac{d}{dt}\left(\int_a^b F\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right) dx\right).$$

Carrying the derivative under the integral sign (which we can do since the integrand $F\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)$ is a continuously differentiable function) gives us that:

$$\frac{d}{dt}(J[\Lambda(x,t)]) = \int_a^b \frac{\partial}{\partial t}\left(F\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\right) dx$$

$$= \int_a^b \left(\frac{\partial F}{\partial y}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\frac{\partial \Lambda(x,t)}{\partial t} + \frac{\partial F}{\partial y'}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\frac{\partial^2 \Lambda}{\partial x \partial t}(x,t)\right) dx.$$

And so, we get that:

Equation 3.6.2: $$\frac{d}{dt}(J[\Lambda(x,t)])$$

$$= \int_a^b \left(\frac{\partial F}{\partial y}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\frac{\partial \Lambda(x,t)}{\partial t} + \frac{\partial F}{\partial y'}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\frac{\partial^2 \Lambda}{\partial x \partial t}(x,t)\right) dx.$$

Looking at the integral in the conclusion of this lemma, we see that there is only a $\frac{\partial \Lambda}{\partial t}$ partial of Λ in the integrand of that integral. So let us get rid of the $\frac{\partial^2 \Lambda}{\partial x \partial t}$ term in the above integral by integrating the $\int_a^b \frac{\partial F}{\partial y'}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\frac{\partial^2 \Lambda}{\partial x \partial t}(x,t) dx$ term by parts. We have that:

$$\int_a^b \frac{\partial F}{\partial y'}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\frac{\partial^2 \Lambda}{\partial x \partial t}(x,t) dx$$

$$= \frac{\partial F}{\partial y'}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\frac{\partial \Lambda(x,t)}{\partial t}\bigg|_{x=a}^{x=b} - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\right)\frac{\partial \Lambda(x,t)}{\partial t} dx.$$

Now, for every fixed $t \in [t_0, t_1]$, $\Lambda(x,t)$ is a curve in the domain of J. This means that $\forall t \in [t_0, t_1]$, the curve $\Lambda(x,t)$ satisfies the boundary conditions:

$$\Lambda(a,t) = A \quad and \quad \Lambda(b,t) = B$$

In other words, for all flow times $t \in [t_0, t_1]$, the values of $\Lambda(x,t)$ on the boundaries: $\Lambda(a,t)$ and $\Lambda(b,t)$ are constant. This means that for any time $t \in (t_0, t_1)$,

$$\frac{\partial \Lambda(a,t)}{\partial t} = \frac{\partial \Lambda(b,t)}{\partial t} = 0.$$

Great! This means that the term

$$\frac{\partial F}{\partial y'}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\frac{\partial \Lambda(x,t)}{\partial t}\Bigg|_{x=a}^{x=b} = 0.$$

And so, we get that our previous integral equation becomes:

$$\int_a^b \frac{\partial F}{\partial y'}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\frac{\partial^2 \Lambda}{\partial x \partial t}(x,t)dx = -\int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\right)\frac{\partial \Lambda(x,t)}{\partial t}dx.$$

Plugging this into equation 3.6.2 finally gives us that:

$$\frac{d}{dt}(J[\Lambda(x,t)])$$
$$= \int_a^b \left(\frac{\partial F}{\partial y}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\left(x, \Lambda(x,t), \frac{\partial \Lambda(x,t)}{\partial x}\right)\right)\right)\frac{\partial \Lambda(x,t)}{\partial t}dx.$$

Using the variational derivative notation this can be rewritten as:

$$\frac{d}{dt}(J[\Lambda(x,t)]) = \int_a^b \frac{\delta J}{\delta y}[\Lambda(x,t)] \cdot \frac{\partial \Lambda(x,t)}{\partial t}dx.$$

This is the equation that we wanted to prove.

■

The equation in the conclusion of the above lemma is very important and it can be used in many contexts in the calculus of variations. Although we haven't really discussed non-linear flows, the above lemma does hold true for more general types of flows other than just linear ones. Indeed, notice that nowhere in the proof of the above lemma did we use that fact that Λ was a "linear" flow. With this lemma in hand we are ready to prove the following theorem.

Theorem 3.6.3: *Let J and K be the functionals:*

$$J[y] = \int_a^b F(x, y, y')dx,$$

$$K[y]=\int_a^b G(x,y,y')dx,$$

where $F, G \in C^2[\mathbb{R}^3]$. Let J and K's domain be the set of curves $y(x) \in C^2[a,b]$ that satisfy the boundary conditions:

$$y(a)=A \quad and \quad y(b)=B$$

where A and B are two real numbers. Suppose that the variational derivatives of J and K are equal everywhere:

$$\frac{\delta J}{\delta y}\equiv\frac{\delta K}{\delta y}.$$

More explicitly: for any h in the domain of J and K,

$$\frac{\delta J}{\delta y}[h]=\frac{\delta K}{\delta y}[h].$$

Suppose also that there exists a curve $h_0(x)$ in the domain of J and K such that J and K are equal at $h_0(x)$:

$$J[h_0(x)]=K[h_0(x)].$$

We can colloquially call $h_0(x)$ the "curve of agreement" of J and K. Then J and K are equal everywhere. More explicitly, for any h in the domain of J and K,

$$J[h(x)]=K[h(x)].$$

Proof: The idea behind the proof of this theorem is exactly the same as the idea behind the proof of the analogous fact with multivariable functions. Take any curve h in the domain of J and K. Now, take the 2-smooth linear flow $\Lambda : [a,b]\times[0,1]\to\mathbb{R}$ defined by:

$$\Lambda(x,t)=h_0(x)(1-t)+h(x)t.$$

Notice that this linear flow travels from the curve $h_0(x)$ to $h(x)$ as t goes from $t=0$ to $t=1$ since:

$$\Lambda(x,0)=h_0(x) \quad and \quad \Lambda(x,1)=h(x).$$

Now, doing a similar sort of calculation as before we have that (here we apply the previous lemma):

$$J[h(x)]=J[\Lambda(x,1)]=(J[\Lambda(x,1)]-J[\Lambda(x,0)])+J[h_0(x)]=\int_0^1\frac{d}{dt}(J[\Lambda(x,t)])dt+J[h_0(x)]$$

$$=\int_0^1\frac{\delta J}{\delta y}[\Lambda(x,t)]\cdot\frac{\partial\Lambda(x,t)}{\partial t}dt+J[h_0(x)]=\int_0^1\frac{\delta K}{\delta y}[\Lambda(x,t)]\cdot\frac{\partial\Lambda(x,t)}{\partial t}dt+K[h_0(x)]$$

$$= \int_0^1 \frac{d}{dt}(K[\Lambda(x,t)])dt + K[h_0(x)] = (K[\Lambda(x,1)] - K[\Lambda(x,0)]) + K[h_0(x)] = K[\Lambda(x,1)]$$

$$= K[h(x)].$$

And so $J[h(x)] = K[h(x)]$. This proves that J and K are equal everywhere.

■

This theorem is significant because in many situations it provides a general way to prove an integral or functional equality equation. It also has the charming property that it resembles the way that we often prove that two multivariable functions are equal everywhere.

There are generalizations of the above theorem. A question that you might consider is: what happens if the domain of a functional is not "convex." Indeed, in the example of the multivariable functions F and G I said to connect the points (x_0, y_0) and (x, y) by the line $l(t)$. But what if the domain of our functions F and G is not the whole of $\mathbb{R}^2$ but some subset that $l(t)$ does not entirely lie in? In calculus class you probably circumvented this difficult by just choosing another curve $\gamma(t)$ that connects the two points and that constantly lies inside of the domain of F and G. A similar sort of thing can happen in the case of functionals and so you'll have to do a similar sort of thing by choosing some non-linear flow that goes between your curve of agreement $h_0(x)$ and $h(x)$. This can lead to some fascinating generalizations of the above theorem.

Perhaps a more obvious consideration of a generalization of the above theorem is its generalization to higher dimensions. Let's briefly look into this question. In the proof of the higher dimensional version of this theorem, you will need the following higher dimensional version of Lemma 3.6.1.

Lemma 3.6.4: *Let $\Omega \subseteq \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ that the Divergence Theorem applies to. Now, let J be a functional defined by (in this lemma, let $\int$ denote an integral over a region in $\mathbb{R}^n$):*

$$J[h(x_1, x_2, \dots, x_n)] = \int_\Omega F\big(x_1, x_2, \dots, x_n, h, h_{x_1}, h_{x_2}, \dots, h_{x_n}\big) \prod_{i=1}^n dx_i$$

where $F \in C^2[\mathbb{R}^{2n+1}]$. Let J's domain be the set of "hypersurfaces" $h(x_1, x_2, \dots, x_n) \in C^2[\Omega]$ that satisfy the boundary conditions:

$$h(x_1, x_2, \dots, x_n) = 0 \quad if \quad (x_1, x_2, \dots, x_n) \in \partial\Omega$$

(in other words, the h's vanishes on $\partial\Omega$). Suppose also that $\Lambda : \Omega \times [t_0, t_1] \to \mathbb{R}$ is a 2-smooth linear flow of hypersurface in the domain of J. Being a flow in the domain of J again merely means that for every fixed flow time $t \in [t_0, t_1]$, $\Lambda(x_1, x_2, \dots, x_n, t)$ is a hypersurface in the domain of J. Then,

$$\frac{d}{dt}(J[\Lambda(x_1, x_2, \dots, x_n, t)]) = \int_a^b \frac{\delta J}{\delta h}[\Lambda(x_1, x_2, \dots, x_n, t)] \cdot \frac{\partial \Lambda(x_1, x_2, \dots, x_n, t)}{\partial t} dt.$$

Or if you understood the inner product interpretation of Lemma 2.4.3 in Chapter Two, then this equation can be rewritten as:

$$\frac{d}{dt}(J[\Lambda(x_1, x_2, \dots, x_n, t)]) = \langle \frac{\delta J}{\delta h}[\Lambda(x_1, x_2, \dots, x_n, t)], \frac{\partial \Lambda(x_1, x_2, \dots, x_n, t)}{\partial t} \rangle.$$

Proof: This is proved similarly as Lemma 3.6.1 except that here you will have to mimic the calculation in the proof of Theorem 2.4.8 after the proof of Lemma 2.4.9. I will leave the details as an exercise to the reader.

■

The following theorem is the higher dimensional version of Theorem 3.6.3:

Theorem 3.6.5: *Let $\Omega \subseteq \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ that the Divergence Theorem applies to. Suppose that Ω is the closure of its interior and that its boundary $\partial\Omega$ is a set of zero Jordan content. Suppose also that we have a function of the form $f : \partial\Omega \to \mathbb{R}$ and that the set:*

$$\{(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n)) \in \mathbb{R}^{n+1} : (x_1, x_2, \dots, x_n) \in \partial\Omega\}$$

can be parametrized by a non-singular C^∞ curve $\gamma(t)$ in $\mathbb{R}^{n+1}$. Now, let J and K be functionals defined by (in this theorem, let $\int$ denote an integral over a region in $\mathbb{R}^n$):

$$J[h(x_1, x_2, \dots, x_n)] = \int_\Omega F(x_1, x_2, \dots, x_n, h, h_{x_1}, h_{x_2}, \dots, h_{x_n}) \prod_{i=1}^{n} dx_i$$

$$K[h(x_1, x_2, \dots, x_n)] = \int_\Omega G(x_1, x_2, \dots, x_n, h, h_{x_1}, h_{x_2}, \dots, h_{x_n}) \prod_{i=1}^{n} dx_i$$

where $F, G \in C^2[\mathbb{R}^{2n+1}]$. Let J and K's domain be the set of "hypersurfaces" $h(x_1, x_2, \dots, x_n) \in C^2[\Omega]$ that satisfy the boundary conditions:

$$h(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) \quad if \quad (x_1, x_2, \dots, x_n) \in \partial\Omega$$

(in other words, the h's intersect γ above $\partial\Omega$). Suppose that the variational derivatives of J and K are equal everywhere:

$$\frac{\delta J}{\delta h} \equiv \frac{\delta K}{\delta h}.$$

More explicitly: for any $\hbar(x_1, x_2, \dots, x_n)$ in the domain of J and K,

$$\frac{\delta J}{\delta h}[\hbar] = \frac{\delta K}{\delta y}[\hbar].$$

Suppose also that there exists a hypersurface $\hbar_0(x_1, x_2, \dots, x_n)$ *in the domain of* J *and* K *such that* J *and* K *are equal at* $\hbar_0(x_1, x_2, \dots, x_n)$*:*

$$J[\hbar_0(x_1, x_2, \dots, x_n)] = K[\hbar_0(x_1, x_2, \dots, x_n)].$$

We can colloquially call $\hbar_0(x_1, x_2, \dots, x_n)$ *the "hypersurface of agreement" of* J *and* K. *Then* J *and* K *are equal everywhere. More explicitly, for any* $\hbar(x_1, x_2, \dots, x_n)$ *in the domain of* J *and* K,

$$J[\hbar(x_1, x_2, \dots, x_n)] = K[\hbar(x_1, x_2, \dots, x_n)].$$

Proof: The proof of this theorem goes exactly the same as the proof of Theorem 3.6.3 except that here you will need to write down more variables. I will leave the details to the reader as an exercise.

■

Section 7: Two Theorems for the Global Gauss-Bonnet Theorem Sections

In the sections on the Global Gauss-Bonnet Theorem in Chapter 5 and Chapter 6 we will need second order versions of Lemma 3.6.4 (second order means that the integrand F of the functional J is a function of second order partials of h as well). In an exercise at the end of Chapter 2 [see future edition of this book] you derive that the Euler-Lagrange partial differential equation for a functional of the form (here I use indexed argument notation and $\int$ denotes an integral over a region in $\mathbb{R}^n$):

$$J[h(x_1, x_2, \dots, x_n)] = \int_\Omega F\left(\{x_k\}_{k=1}^n, h, \left\{h_{x_k}\right\}_{k=1}^n, \left\{\left\{h_{x_k x_j}\right\}_{k=1}^n\right\}_{j=k}^n\right) \prod_{i=1}^n dx_i,$$

where the domain of J is the same as the domain of the functional in Theorem 2.4.8, is given by:

$$\frac{\partial F}{\partial h} - \sum_{k=1}^n \frac{\partial}{\partial x_k}\left(\frac{\partial F}{\partial h_{x_k}}\right) + \sum_{k=1}^n \sum_{j=k}^n \frac{\partial^2}{\partial x_k \partial x_j}\left(\frac{\partial F}{\partial h_{x_k x_j}}\right) = 0.$$

As a result, we defined the variational derivative of such a functional J as:

$$\frac{\delta J}{\delta h} = \frac{\partial F}{\partial h} - \sum_{k=1}^n \frac{\partial}{\partial x_k}\left(\frac{\partial F}{\partial h_{x_k}}\right) + \sum_{k=1}^n \sum_{j=k}^n \frac{\partial^2}{\partial x_k \partial x_j}\left(\frac{\partial F}{\partial h_{x_k x_j}}\right).$$

In the special case $n = 2$ this functional, its Euler-Lagrange partial differential equation, and variational derivative are respectively given by:

$$J[h(x,y)] = \int_\Omega F\left(x, y, h, h_x, h_y, h_{xx}, h_{xy}, h_{yy}\right) dx dy,$$

$$\frac{\partial F}{\partial h}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial h_{xx}}\right)+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial h_{xy}}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial h_{yy}}\right)=0,$$

$$\frac{\delta J}{\delta h}=\frac{\partial F}{\partial h}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial h_{xx}}\right)+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial h_{xy}}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial h_{yy}}\right).$$

With this more general form of the variational derivative we can now obtain a more general form of Lemma 3.6.4 that we will need for the sections on the Global Gauss-Bonnet Theorem. I will give the statement of the lemmas in both the $n = 2$ case and the general case. In this section we are going to let everything be C^∞ because in the Gauss-Bonnet Theorem we will be working with infinitely differentiable surfaces. This adds no difficulties or technicalities to the following theorems or their proofs.

Theorem 3.7.1: *Let $\Omega \subseteq \mathbb{R}^2$ be a compact subset of $\mathbb{R}^2$ that the Divergence Theorem[21] applies to. Suppose that Ω is the closure of its interior and that its boundary $\partial\Omega$ is a set of zero Jordan content. Suppose also that we have a function of the form $f : \partial\Omega \to \mathbb{R}$ and that the set:*

$$\{(x, y, f(x, y)) \in \mathbb{R}^3 : (x, y) \in \partial\Omega\}$$

can be parametrized by a non-singular C^∞ curve $\gamma(t)$ in $\mathbb{R}^3$. Now, let J be a functional of the form:

$$J[h(x,y)] = \iint_\Omega F(x, y, h, h_x, h_y, h_{xx}, h_{xy}, h_{yy})dxdy$$

where $F \in C^\infty[\mathbb{R}^8]$. Let J's domain be the set of "surfaces" $h(x, y) \in C^\infty[\Omega]$ that satisfy the boundary conditions:

$$h(x, y) = f(x, y) \quad if \quad (x, y) \in \partial\Omega$$

(in other words, the h's intersect γ above $\partial\Omega$). Suppose also that $\Lambda : \Omega \times [t_0, t_1] \to \mathbb{R}$ is an ∞-smooth flow of surfaces in the domain of J. Being a flow in the domain of J again merely means that for every fixed flow time $t \in [t_0, t_1]$, $\Lambda(x, y, t)$ is a surface in the domain of J. Suppose also the flow Λ satisfies the boundary conditions that its value and all of its first partials are constant/unchanging on $\partial\Omega$ for all time:

$$\forall (x, y) \in \partial\Omega, \qquad \frac{d}{dt}\big(\Lambda(x, y)\big) = \frac{\partial \Lambda}{\partial t}(x, y) \equiv 0$$

$$\forall (x, y) \in \partial\Omega, \qquad \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial x}(x, y)\right) = \frac{\partial^2 \Lambda}{\partial x \partial t}(x, y) \equiv 0,$$

[21] One could totally use Green's Theorem here since Ω is in $\mathbb{R}^2$. But I decided to keep things easily generalizable here by just using the two-dimensional version of the Divergence Theorem (which is in fact completely equivalent to Green's Theorem up to rephrasing).

$$\forall (x,y) \in \partial\Omega, \qquad \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial y}(x,y)\right) = \frac{\partial^2 \Lambda}{\partial y \partial t}(x,y) \equiv 0,$$

on all of $t \in [t_0, t_1]$. *Then,*

$$\frac{d}{dt}(J[\Lambda(x,y,t)]) = \iint_\Omega \frac{\delta J}{\delta h}[\Lambda(x,y,t)] \cdot \frac{\partial \Lambda(x,y,t)}{\partial t} dxdy.$$

Or if you understood the inner product interpretation of Lemma 2.4.2 in Chapter Two, then this equation can be rewritten as:

$$\frac{d}{dt}(J[\Lambda(x,y,t)]) = \langle \frac{\delta J}{\delta h}[\Lambda(x,y,t)], \frac{\partial \Lambda(x,y,t)}{\partial t} \rangle.$$

Proof: The proof of this theorem is again just a calculation that involves the Divergence Theorem, except that here we will have to apply the Divergence Theorem more than once. We have that (I will omit writing the partials of Λ here):

$$\frac{d}{dt}(J[\Lambda]) = \frac{d}{dt}\left(\iint_\Omega F\big(x,y,\Lambda,\Lambda_x,\Lambda_y,\Lambda_{xx},\Lambda_{xy},\Lambda_{yy}\big)dxdy\right).$$

Carrying the derivative under the integral sign (which we can do since the integrand is continuously differentiable and the domain of integration is compact) gives us that (after the second term below I will omit writing the argument of F; it is being evaluated at $\big(x,y,\Lambda,\Lambda_x,\Lambda_y,\Lambda_{xx},\Lambda_{xy},\Lambda_{yy}\big)$):

$Equation$ 3.7.2: $$\frac{d}{dt}(J[\Lambda]) = \iint_\Omega \frac{\partial}{\partial t}\Big(F\big(x,y,\Lambda,\Lambda_x,\Lambda_y,\Lambda_{xx},\Lambda_{xy},\Lambda_{yy}\big)\Big)\, dxdy$$

$$= \iint_\Omega \left(\frac{\partial F}{\partial h}\Lambda_t + \frac{\partial F}{\partial h_x}\Lambda_{xt} + \frac{\partial F}{\partial h_y}\Lambda_{yt} + \frac{\partial F}{\partial h_{xx}}\Lambda_{xxt} + \frac{\partial F}{\partial h_{xy}}\Lambda_{xyt} + \frac{\partial F}{\partial h_{yy}}\Lambda_{yyt}\right) dxdy.$$

Let's get rid of all of the x and y partials of the Λ's. Let's start doing this by taking a look at the $\iint_\Omega \left(\frac{\partial F}{\partial h_{xx}}\Lambda_{xxt} + \frac{\partial F}{\partial h_{xy}}\Lambda_{xyt} + \frac{\partial F}{\partial h_{yy}}\Lambda_{yyt}\right) dxdy$ term in the above integral and "integrating it by parts." "Integrating this term by parts" gives that:

$$\iint_\Omega \left(\frac{\partial F}{\partial h_{xx}}\Lambda_{xxt} + \frac{\partial F}{\partial h_{xy}}\Lambda_{xyt} + \frac{\partial F}{\partial h_{yy}}\Lambda_{yyt}\right) dxdy$$

$$= \iint_\Omega \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_{xx}}\Lambda_{xt}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_{xy}}\Lambda_{xt} + \frac{\partial F}{\partial h_{yy}}\Lambda_{yt}\right)\right) dxdy$$

$$- \iint_\Omega \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_{xx}}\right)\Lambda_{xt} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_{xy}}\right)\Lambda_{xt} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_{yy}}\right)\Lambda_{yt}\right) dxdy.$$

Applying the Divergence Theorem to the first integral on the right-hand side gives that:

$$\iint_{\Omega} \left(\frac{\partial F}{\partial h_{xx}}\Lambda_{xxt} + \frac{\partial F}{\partial h_{xy}}\Lambda_{xyt} + \frac{\partial F}{\partial h_{yy}}\Lambda_{yyt}\right) dxdy$$

$$= \oint_{\partial\Omega} \left(\frac{\partial F}{\partial h_{xx}}\Lambda_{xt}, \frac{\partial F}{\partial h_{xy}}\Lambda_{xt} + \frac{\partial F}{\partial h_{yy}}\Lambda_{yt}\right) \cdot d\vec{n}$$

$$- \iint_{\Omega} \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_{xx}}\right)\Lambda_{xt} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_{xy}}\right)\Lambda_{xt} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_{yy}}\right)\Lambda_{yt}\right) dxdy.$$

We have that Λ_{xt} and Λ_{yt} are constantly zero on the boundary and so the first integral on the right-hand side is zero! Thus we get that:

$$\iint_{\Omega} \left(\frac{\partial F}{\partial h_{xx}}\Lambda_{xxt} + \frac{\partial F}{\partial h_{xy}}\Lambda_{xyt} + \frac{\partial F}{\partial h_{yy}}\Lambda_{yyt}\right) dxdy$$

$$= - \iint_{\Omega} \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_{xx}}\right)\Lambda_{xt} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_{xy}}\right)\Lambda_{xt} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_{yy}}\right)\Lambda_{yt}\right) dxdy.$$

Integrating the integral on the right-hand side "by parts again" gives that:

$$\iint_{\Omega} \left(\frac{\partial F}{\partial h_{xx}}\Lambda_{xxt} + \frac{\partial F}{\partial h_{xy}}\Lambda_{xyt} + \frac{\partial F}{\partial h_{yy}}\Lambda_{yyt}\right) dxdy$$

$$= -(\iint_{\Omega} \left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_{xx}}\right)\Lambda_t + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_{xy}}\right)\Lambda_t\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_{yy}}\right)\Lambda_t\right)\right) dxdy$$

$$- \iint_{\Omega} \left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial h_{xx}}\right)\Lambda_t + \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial F}{\partial h_{xy}}\right)\Lambda_t + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial h_{yy}}\right)\Lambda_t\right) dxdy).$$

Applying the Divergence Theorem again to the first integral in the right-hand side of the above equation gives that:

$$\iint_{\Omega} \left(\frac{\partial F}{\partial h_{xx}}\Lambda_{xxt} + \frac{\partial F}{\partial h_{xy}}\Lambda_{xyt} + \frac{\partial F}{\partial h_{yy}}\Lambda_{yyt}\right) dxdy$$

$$= - \oint_{\partial\Omega} \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_{xx}}\right)\Lambda_t + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_{xy}}\right)\Lambda_t, \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_{yy}}\right)\Lambda_t\right) d\vec{n} +$$

$$\iint_{\Omega} \left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial h_{xx}}\right)\Lambda_t + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial h_{xy}}\right)\Lambda_t + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial h_{yy}}\right)\Lambda_t\right) dxdy.$$

We have that Λ_t zero on the boundary and so the first integral on the right-hand side is zero! So, we get that:

$$Equation\ 3.7.3: \qquad \iint\limits_{\Omega} \left(\frac{\partial F}{\partial h_{xx}}\Lambda_{xxt} + \frac{\partial F}{\partial h_{xy}}\Lambda_{xyt} + \frac{\partial F}{\partial h_{yy}}\Lambda_{yyt}\right)dxdy$$

$$= \iint\limits_{\Omega} \left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial h_{xx}}\right)\Lambda_t + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial h_{xy}}\right)\Lambda_t + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial h_{yy}}\right)\Lambda_t\right)dxdy.$$

Yes! We've gotten rid of the x and y partials of the Λ in this term using the Divergence Theorem. Now let's look at the $\iint_\Omega \left(\frac{\partial F}{\partial h_x}\Lambda_{xt} + \frac{\partial F}{\partial h_y}\Lambda_{yt}\right)dxdy$ term in the integral in Equation 3.7.2. "Integrating this term by parts" gives that:

$$\iint\limits_{\Omega} \left(\frac{\partial F}{\partial h_x}\Lambda_{xt} + \frac{\partial F}{\partial h_y}\Lambda_{yt}\right)dxdy$$

$$= \iint\limits_{\Omega} \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\Lambda_t\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\Lambda_t\right)\right)dxdy - \iint\limits_{\Omega} \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right)\Lambda_t + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right)\Lambda_t\right)dxdy.$$

Applying the Divergence Theorem to the first integral on the right-hand side gives that:

$$\iint\limits_{\Omega} \left(\frac{\partial F}{\partial h_x}\Lambda_{xt} + \frac{\partial F}{\partial h_y}\Lambda_{yt}\right)dxdy$$

$$= \oint\limits_{\partial\Omega} \left(\frac{\partial F}{\partial h_x}\Lambda_t, \frac{\partial F}{\partial h_y}\Lambda_t\right)\cdot d\vec{n} - \iint\limits_{\Omega} \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right)\Lambda_t + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right)\Lambda_t\right)dxdy.$$

We have that Λ_t is constantly zero on the boundary $\partial\Omega$ and so the first integral on the right-hand side is zero. So we have that:

$$\iint\limits_{\Omega} \left(\frac{\partial F}{\partial h_x}\Lambda_{xt} + \frac{\partial F}{\partial h_y}\Lambda_{yt}\right)dxdy = -\iint\limits_{\Omega} \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right)\Lambda_t + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right)\Lambda_t\right)dxdy.$$

So we've gotten rid of all of the x and y partials of Λ here as well. Plugging this Equation and Equation 3.7.3 back into Equation 3.7.2 finally gives us that:

$$\frac{d}{dt}(J[\Lambda])$$

$$= \iint\limits_{\Omega} \left(\frac{\partial F}{\partial h} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial h_{xx}}\right) + \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial F}{\partial h_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial h_{yy}}\right)\right)\Lambda_t dxdy.$$

Which is in fact equivalent to the equation that we wanted to prove:

$$\frac{d}{dt}(J[\Lambda]) = \iint\limits_{\Omega} \frac{\delta J}{\delta h}[\Lambda] \cdot \Lambda_t dxdy.$$

■

The general version of the above lemma goes as the following.

Theorem 3.7.4: *Let $\Omega \subseteq \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ that the Divergence Theorem applies to. Suppose that Ω is the closure of its interior and that its boundary $\partial\Omega$ is a set of zero Jordan content. Suppose also that we have a function of the form $f : \partial\Omega \to \mathbb{R}$ and that the set:*

$$\{(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n)) \in \mathbb{R}^{n+1} : (x_1, x_2, \dots, x_n) \in \partial\Omega\}$$

can be parametrized by a non-singular C^∞ curve $\gamma(t)$ in $\mathbb{R}^{n+1}$. Now, let J be a functional defined by (in this theorem, let $\int$ denote an integral over a region in $\mathbb{R}^n$):

$$J[h(x_1, x_2, \dots, x_n)] = \int\limits_{\Omega} F\left(\{x_k\}_{k=1}^n, h, \{h_{x_k}\}_{k=1}^n, \left\{\{h_{x_k x_j}\}_{k=1}^n\right\}_{j=k}^n\right) \prod_{i=1}^n dx_i$$

where $F \in C^\infty[\mathbb{R}^{2n+1+n(n-1)/2}]$. Let J's domain be the set of "hypersurfaces" $h(x_1, x_2, \dots, x_n) \in C^\infty[\Omega]$ that satisfy the boundary conditions:

$$h(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) \quad if \quad (x_1, x_2, \dots, x_n) \in \partial\Omega$$

(in other words, the h's pass through γ above $\partial\Omega$). Suppose also that $\Lambda : \Omega \times [t_0, t_1] \to \mathbb{R}$ is an ∞-smooth flow of surfaces in the domain of J. Being a flow in the domain of J again merely means that for every fixed flow time $t \in [t_0, t_1]$, $\Lambda(x_1, x_2, \dots, x_n, t)$ is a hypersurface in the domain of J. Suppose also the flow Λ satisfies the boundary conditions that its value and all of its first partials are constant/unchanging on $\partial\Omega$ for all time:

$$\forall (x_1, x_2, \dots, x_n) \in \partial\Omega, \qquad \frac{d}{dt}\left(\Lambda(x_1, x_2, \dots, x_n)\right) = \frac{\partial \Lambda}{\partial t}(x_1, x_2, \dots, x_n) \equiv 0$$

$$\forall k \in \{1,2, \dots, n\} \quad \forall (x_1, x_2, \dots, x_n) \in \partial\Omega,$$

$$\frac{d}{dt}\left(\frac{\partial \Lambda}{\partial x_k}(x_1, x_2, \dots, x_n)\right) = \frac{\partial^2 \Lambda}{\partial x_k \partial t}(x_1, x_2, \dots, x_n) \equiv 0,$$

on all of $t \in [t_0, t_1]$. *Then,*

$$\frac{d}{dt}(J[\Lambda(x_1, x_2, \dots, x_n, t)]) = \int\limits_{\Omega} \frac{\delta J}{\delta h}[\Lambda(x_1, x_2, \dots, x_n, t)] \cdot \frac{\partial \Lambda(x_1, x_2, \dots, x_n, t)}{\partial t} \prod_{k=1}^n dx_k.$$

Or if you understood the inner product interpretation of Lemma 2.4.3 in Chapter Two, then this equation can be rewritten as:

$$\frac{d}{dt}(J[\Lambda(x_1, x_2, \dots, x_n, t)]) = \langle \frac{\delta J}{\delta h}[\Lambda(x_1, x_2, \dots, x_n, t)], \frac{\partial \Lambda(x_1, x_2, \dots, x_n, t)}{\partial t} \rangle.$$

Proof: The proof of this theorem is exactly the same as the proof of the previous theorem except that here you will have to deal with more variables. It is again just a calculation using the Divergence Theorem several times. I will leave the details to the reader.

■

For the linear algebra calculation in the section on the Global Gauss-Bonnet Theorem in Chapter 6 however, it will turn out to be more convenient to consider quantities such as h_{xy} and h_{yx}, even though they are equal, as two different things purely for summation indexing reasons. So in the following version of the above theorem we are going to let the integrand F have argument space for both $h_{x_k x_j}$ and $h_{x_j x_k}$ for any $k, j \in \{1,2,\dots,n\}$. This is all done for convenience when we deal with the Hessian matrix in the Global Gauss-Bonnet Theorem section in Chapter 6.

Theorem 3.7.5: *Let $\Omega \subseteq \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ that the Divergence Theorem applies to. Suppose that Ω is the closure of its interior and that its boundary $\partial\Omega$ is a set of zero Jordan content. Suppose also that we have a function of the form $f : \partial\Omega \to \mathbb{R}$ and that the set:*

$$\left\{\left(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n)\right) \in \mathbb{R}^{n+1} : (x_1, x_2, \dots, x_n) \in \partial\Omega\right\}$$

can be parametrized by a non-singular C^∞ curve $\gamma(t)$ in $\mathbb{R}^{n+1}$. Now, let J be a functional defined by (in this theorem, let $\int$ denote an integral over a region in $\mathbb{R}^n$):

$$J[h(x_1, x_2, \dots, x_n)] = \int_\Omega F\left(\{x_k\}_{k=1}^n, h, \left\{h_{x_k}\right\}_{k=1}^n, \left\{\left\{h_{x_k x_j}\right\}_{k=1}^n\right\}_{j=1}^n\right) \prod_{i=1}^n dx_i$$

<u>(note that the sum for j now starts at 1 instead of k like in the previous theorem)</u> where $F \in C^\infty\left[\mathbb{R}^{2n+1+n^2}\right]$. Let J's domain be the set of "hypersurfaces" $h(x_1, x_2, \dots, x_n) \in C^\infty[\Omega]$ that satisfy the boundary conditions:

$$h(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) \quad if \quad (x_1, x_2, \dots, x_n) \in \partial\Omega$$

(in other words, the h's pass through γ above $\partial\Omega$). Suppose also that $\Lambda : \Omega \times [t_0, t_1] \to \mathbb{R}$ is an ∞-smooth flow of surfaces in the domain of J. Being a flow in the domain of J again merely means that for every fixed flow time $t \in [t_0, t_1]$, $\Lambda(x_1, x_2, \dots, x_n, t)$ is a hypersurface in the domain of J. Suppose also the flow Λ satisfies the boundary conditions that its value and all of its first partials are constant/unchanging on $\partial\Omega$ for all time:

$$\forall (x_1, x_2, \dots, x_n) \in \partial\Omega, \qquad \frac{d}{dt}\left(\Lambda(x_1, x_2, \dots, x_n)\right) = \frac{\partial\Lambda}{\partial t}(x_1, x_2, \dots, x_n) \equiv 0$$

$$\forall k \in \{1,2,\dots,n\} \ \ \forall (x_1, x_2, \dots, x_n) \in \partial\Omega,$$

$$\frac{d}{dt}\left(\frac{\partial\Lambda}{\partial x_k}(x_1, x_2, \dots, x_n)\right) = \frac{\partial^2\Lambda}{\partial x_k \partial t}(x_1, x_2, \dots, x_n) \equiv 0,$$

on all of $t \in [t_0, t_1]$. *Then,*

$$\frac{d}{dt}(J[\Lambda(x_1,x_2,\dots,x_n,t)]) = \int_\Omega \frac{\delta J}{\delta h}[\Lambda(x_1,x_2,\dots,x_n,t)] \cdot \frac{\partial\Lambda(x_1,x_2,\dots,x_n,t)}{\partial t}\prod_{k=1}^{n} dx_k.$$

where the functional derivative $\frac{\delta J}{\delta h}$ *here is given by:*

$$\frac{\delta J}{\delta h} = \frac{\partial F}{\partial h} - \sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial F}{\partial h_{x_k}}\right) + \sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_k \partial x_j}\left(\frac{\partial F}{\partial h_{x_k x_j}}\right).$$

And of course, the previous equation does have an inner product interpretation as always.

Proof: This is proved just like the previous theorem except that here you will have to do a bit more diverse set of "integration by parts" via the Divergence Theorem.

■

Chapter 4: Introduction to Differential Geometry

"I've always felt just a tiny bit guilty about it. Not enough to stop me, of course – I'll reverse the order of integration on a double integral as fast as you can snap your fingers. I pin my fate (perhaps foolishly) on the hope that, as one mathematician put it, 'In mathematics, as in life, virtue is not always rewarded, *nor vice always punished*' (my emphasis)" – Paul J. Nahin.

"[Haim], don't glorify such calculations" – Professor Morrow

"Umm…" – Haim

Section 1: Outline

The first half of this book was devoted to the study of the calculus of variations and now we shall transition to the study of differential geometry. Differential geometry, colloquially speaking, is the study of smooth surfaces where "smooth" means that the surface curves "nicely" everywhere and doesn't have any bad edges or turns. Surfaces appear all over the place and they are a part of our everyday lives. Our experiences with surfaces include walking on the surface of our planet, holding a beach ball, admiring the curvature of the domes on the Taj Mahal, etc. Since surfaces are so ingrained in the everyday lives of human beings, it's no surprise whatsoever that mathematicians have taken an interest in studying them and their properties. This gave rise to the field called differential geometry.

And it turns out that the study of surfaces and their more general forms called "manifolds" have important application in other fields of study such as physics. One such famous example is their use in the General Theory of Relativity where space-time is described as a 4-dimensional curved manifold. It's cool to consider the fact that the mathematics that we are about to cover in this and in the next few chapters is very similar to the mathematics used in the General Theory of Relativity.[22]

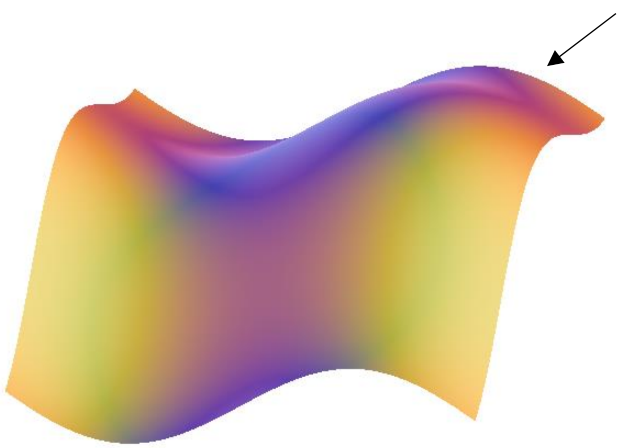
A great number of questions that come up in the study of surfaces are variational in nature. The most classical of such examples is the question of what is the shortest path between two points on a surface (the minimizing curve problem). As a result, the calculus of variations plays a very significant and crucial role in differential geometry. In the next few chapters we will explore the applications that the calculus of variations that we developed in the past three chapters has to theory of surfaces and more general manifolds. The material that we will cover from henceforth in this book is called "variational differential geometry."

This chapter focuses on getting the reader acquainted with all of the basic definitions, notations, and theorems of differential geometry. If you already studied differential geometry, then you might consider skimming this chapter to make sure that you know everything and then go on to the next chapter.

I want to note that pretty much all of the introduction to differential geometry in this chapter, unless stated otherwise, comes from P. Do Carmo's textbook on differential geometry (see bibliography).

Section 2: Definition of a Smooth Surface

The first task in the study of smooth surfaces should of course be the business of defining what smooth surfaces are. There are two main and equivalent ways of going about this. The first way is to define a smooth surface as a set that locally at any point is the graph of a smooth function. Indeed, in calculus class you were probably introduced multiple times to the idea of representing a surface as the graph of a function $z = f(x, y)$.

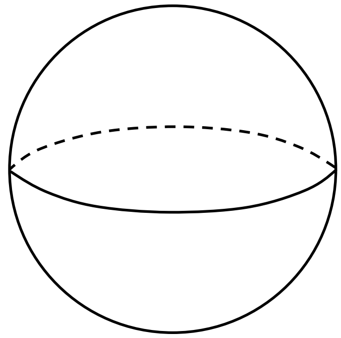


A surface that is the graph of a function $f(x, y)$.

Using the graph of a function to describe a surface is quite natural. However, the problem is that not all surfaces, such as the sphere, can be represented the graph of a single function. For

[22] A topic which the author admittingly does not know anything about yet.

example, in the case of the sphere notice that there is no way to rotate the sphere so that it becomes the graph of some function $f(x,y)$.

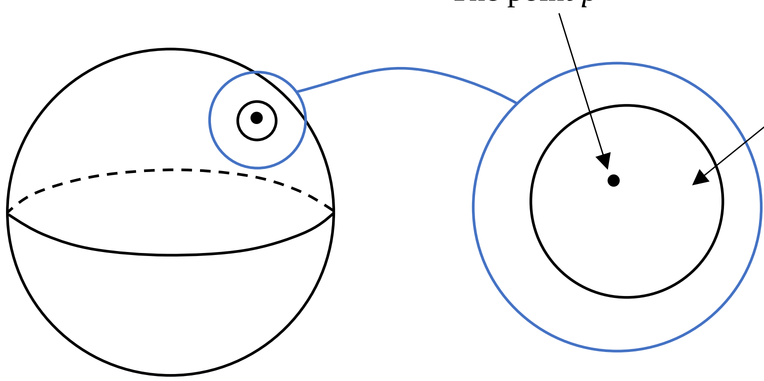


So how can we describe such surfaces that don't yield to graph representations by any one single function? The idea is: well if we can't describe the surface as the graph of one function, let's describe it as the graph of different functions in different places. With this we come to our first definition of a smooth surface.

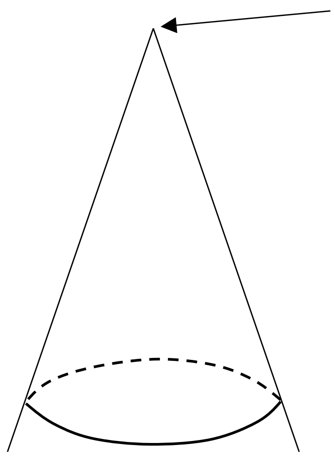
Definition 4.2.1 (First Definition of a Smooth Two-Dimensional Surface Sitting in $\mathbb{R}^3$): *A set $S \subseteq \mathbb{R}^3$ is called a **smooth two-dimensional surface** sitting in $\mathbb{R}^3$ if it satisfies the following property: for any point $p \in S$, locally to p S is the graph of a $C^\infty[\mathbb{R}^2]$ function over some open set. Stated more precisely, S is a smooth two-dimensional surface if for any $p \in S$ there exists an open set V containing p and a function of the form $z = f(x,y)$, $x = f(y,z)$, or $y = f(x,z)$ where $f \in C^\infty[U]$ and U is some open set in $\mathbb{R}^2$ such that $V \cap S$ is the graph of f over U.*

The above is merely a technical definition that just states that a surface is a set that locally to any point on it is the graph of an infinitely differentiable function $f : U \to \mathbb{R}$ over some small open set U. So instead of representing an entire surface globally as the graph of a smooth function, the above definition only requires that we can represent the surface locally to any point as the graph of a smooth function. Here locality to the point on the surface is given by open neighborhood V of p. The condition C^∞ in the above definition of a smooth surface can be relaxed to C^3 for the purposes of the theory that we will be developing in the next few chapters. But let us stick with convention by requiring it to be C^∞.

Let us take another look at the example of the sphere. Notice that locally to any point of a sphere of radius $r > 0$, we can describe the surface as the graph of one of the following 6 functions: $z = \sqrt{r^2 - x^2 - y^2}$, $z = -\sqrt{r^2 - x^2 - y^2}$, $x = \sqrt{r^2 - y^2 - z^2}$, $x = -\sqrt{r^2 - y^2 - z^2}$, $y = \sqrt{r^2 - x^2 - z^2}$, $y = -\sqrt{r^2 - x^2 - z^2}$.



So a sphere is a smooth two-dimensional surface. Other examples of smooth two-dimensional surfaces include many surfaces that we've encountered in life such as: planes, ellipsoid, hyperboloids, paraboloids, etc. An example of a non-smooth surface is the cone since at the tip of the cone it's impossible to locally describe the surface as the graph of a C^∞ function.



Representing a surface as the graph of a function is a very classical and convenient technique for parametrizing a surface. However there are instances when it's possible or more convenient to use a different parametrization of a surface. For example the sphere of radius $r > 0$, alongside the local function graph representation, can also be famously parametrized in spherical coordinates. In the spherical coordinates representation what one does is they take any point (x, y, z) on the sphere, let θ be the angle that (x, y) makes with the x-axis, let φ be the angle that (x, y, z) makes with the z-axis, and then by geometric reasons it's not too hard to see that:

$$(x, y, z) = (r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$$

where r is the radius of the sphere. So spherical coordinates provide us with a different way to represent the surface of the sphere by giving us a new coordinate system to represent points on the surface. And this situation where the surface admits a non-graph like parametrization (such as spherical coordinates for the sphere) happens all the time in differential geometry and in many cases these other parametrizations are much easier to work with rather than looking at the surface as the graph of a function locally. In fact, the existence of a local graph representation of a surface may sometimes be only accessible through a theoretical application of the implicit function theorem and so more general types of parametrizations of surfaces are sometimes the only way to have a handle on a specific function that generates a surface. As a result, the ability to represent surfaces by more complicated parametrizations is crucial and people have done this by defining surfaces as the local image of what is called "surface parametrizations." With this we arrive out our alternative (and we will show equivalent) definition of a surface.

Definition 4.2.2 (Second Definition of a Smooth Two-Dimensional Surface Sitting in $\mathbb{R}^3$): *A set $S \subseteq \mathbb{R}^3$ is called a* ***smooth two-dimensional surface*** *sitting in $\mathbb{R}^3$ if for every point $p \in S$ on the surface, there exists a surface parametrization of S located at p. Now let's define what a surface parametrization is. A* ***surface parametrization*** *of a surface S is a function of the form $\Phi : U \subseteq \mathbb{R}^2 \to V \cap S$ where U and V are open sets that satisfy the following three properties:*

1.) $\Phi(u,v) = \left(\Phi_x(u,v), \Phi_y(u,v), \Phi_z(u,v)\right)$ is infinitely differentiable. Symbolically this is written as $\Phi \in C^\infty[U, \mathbb{R}]$ or more explicitly $\Phi_x, \Phi_y, \Phi_z \in C^\infty[U]$. This condition is included so that we will be able to take lots of derivatives of the surface.

2.) Φ is a homeomorphism. This means that Φ is a bijection between the sets U and $V \cap S$ and that both Φ and Φ^{-1} (the inverse of Φ) are continuous functions. The continuity of Φ is actually already given by the first property. This condition is important because the bijective property of Φ is what justifies the phrase that the surface parametrization Φ "parametrizes" the surface S locally. For those who know topology, the fact that Φ is a homeomorphism implies that the surface S is locally topologically similar to $\mathbb{R}^2$. Indeed, a surface can be thought of as a "bent" version of the plane $\mathbb{R}^2$.

3.) The differential of Φ has maximal rank everywhere. In other words, for every point $(u,v) \in U$ in the domain of Φ, the differential matrix of Φ:

$$D\Phi(u,v) = \begin{bmatrix} \frac{\partial \Phi_x}{\partial u}(u,v) & \frac{\partial \Phi_x}{\partial v}(u,v) \\ \frac{\partial \Phi_y}{\partial u}(u,v) & \frac{\partial \Phi_y}{\partial v}(u,v) \\ \frac{\partial \Phi_z}{\partial u}(u,v) & \frac{\partial \Phi_z}{\partial v}(u,v) \end{bmatrix}$$

has maximal rank ("maximal rank" means that some 2 by 2 submatrix of the above 3 by 2 matrix has non-zero determinant). An equivalent way to state this is that the following Jacobians:

$$\frac{\partial\left(\Phi_x, \Phi_y\right)}{\partial(u,v)}, \frac{\partial(\Phi_x, \Phi_z)}{\partial(u,v)}, \frac{\partial\left(\Phi_y, \Phi_z\right)}{\partial(u,v)}$$

don't ever vanish simultaneously on U (these are in fact the 2 by 2 submatrices of the above 3 by 2 matrix). Another equivalent way to state this is that the columns of the $D\Phi(u,v)$ matrix are always linearly independent. Symbolically this can be written as:

$$\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}=\begin{bmatrix}\frac{\partial\Phi_x}{\partial u}(u,v)\\ \frac{\partial\Phi_y}{\partial u}(u,v)\\ \frac{\partial\Phi_z}{\partial u}(u,v)\end{bmatrix}\times\begin{bmatrix}\frac{\partial\Phi_x}{\partial v}(u,v)\\ \frac{\partial\Phi_y}{\partial v}(u,v)\\ \frac{\partial\Phi_z}{\partial v}(u,v)\end{bmatrix}\neq 0$$

on $(u,v)\in U$ where $\times$ denotes the vector cross product. The equivalence of these three ways to state this condition merely follows from the definitions of the notations involved. This condition is important for several reasons one of which is that it allows us to easily talk about the tangent plane to the surface (the columns of $D\Phi$ span the tangent plane in fact).

The purpose of U and $V\cap S$ is that they create small neighborhoods in the domain and range of the function Φ so that we have "space" to differentiate and talk about continuity of Φ and Φ^{-1} in the classical sense. Now we say that the surface parametrization Φ is "located at $p\in S$" (or simply "at p") if $p\in\mathrm{ran}(\Phi)$. So as already stated above, S is a surface if at every point $p\in S$ on the surface there exists a surface parametrization Φ of S located at p.

The above definition is similar in nature to the first definition of a surface (Definition 4.2.1) in that it uses a structure to describe a smooth surface locally. However the tool that it uses to describe the surface locally are not graphs of functions, but parametrizations called "surface parametrizations." Surface parametrizations are often more convenient to work with and handle and so the above definition is usually taken to be the starting point for the study of smooth surfaces. However both of the above definitions of a surface are equivalent and we will use the perspectives provided by both of them in our study of differential geometry. Before we prove the equivalence of Definitions 4.2.1 and 4.2.2, let us look at some examples of surface parametrizations.

One of the things that differential geometry students quickly discover when they first start working with surface parametrizations is that it is often really hard and/or tedious to prove that the inverse of a surface parametrization is continuous. However, in those situation when you already know that S is a regular surface and you just want to prove that Φ is a surface parametrization of S, it turns out that it is sufficient to just check that it is bijective and that it satisfies the first and third conditions of a chart. In other words, it isn't necessary to check that Φ^{-1} is continuous. This often saves a lot of trouble because proving that Φ^{-1} is continuous is often a difficult task. This fact will be the subject of Theorem 4.2.6. But first let us look at some specific examples of surface parametrizations.

Example 4.2.3: Let's return to the example of the spherical coordinate representation of a sphere. Let $S_r^{[2]}$ denote the sphere of radius r and let us take the function Φ defined by:

$$\Phi(\theta,\varphi) = (r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$$

where $\text{dom}(\Phi) = (0, 2\pi) \times (0, \pi)$. This function is the spherical coordinates representation of points on the sphere that we encountered earlier. Let's check that Φ is a surface parametrization of this sphere. Φ does map into this sphere since as you can check with the above equation $\|\Phi(\theta,\varphi)\|$ is always equal to r. Φ is clearly infinitely differentiable and thus satisfies the first condition of a surface parametrization. By looking at the geometric considerations for how the spherical coordinates formula was created, it shouldn't be hard to see that Φ is a bijection between $\text{dom}(\Phi)$ and $\text{ran}(\Phi)$ (although this can easily be proven directly, try it!). And since we showed above that the sphere is a smooth surface by showing that it is locally representable as the graph of C^∞ functions, we get by Theorem 4.2.6 below that Φ^{-1} is also continuous (we're technically also using the fact here that smooth surface as defined in Definition 4.2.1 and 4.2.2 are equivalent, which we will prove in Theorem 4.2.8 below). I do want to note that in this case it isn't too difficult to show that Φ^{-1} is continuous. So Φ is a homeomorphism.

Now let us check that Φ satisfies the 3rd condition of a surface parametrization. Calculating all of the Jacobians of Φ gives that:

$$\frac{\partial(\Phi_x,\Phi_y)}{\partial(\theta,\varphi)} = \det\left(\begin{bmatrix} \frac{\partial\Phi_x}{\partial\theta}(\theta,\varphi) & \frac{\partial\Phi_x}{\partial\varphi}(\theta,\varphi) \\ \frac{\partial\Phi_y}{\partial\theta}(\theta,\varphi) & \frac{\partial\Phi_y}{\partial\varphi}(\theta,\varphi) \end{bmatrix}\right) = \det\left(\begin{bmatrix} -r\sin(\varphi)\sin(\theta) & r\cos(\varphi)\cos(\theta) \\ r\sin(\varphi)\cos(\theta) & r\cos(\varphi)\sin(\theta) \end{bmatrix}\right)$$

$$= -r^2\cos(\varphi)\sin(\varphi)\left(\sin^2(\theta) + \cos^2(\theta)\right) = -r^2\cos(\varphi)\sin(\varphi),$$

$$\frac{\partial(\Phi_x,\Phi_z)}{\partial(\theta,\varphi)} = \det\left(\begin{bmatrix} \frac{\partial\Phi_x}{\partial\theta}(\theta,\varphi) & \frac{\partial\Phi_x}{\partial\varphi}(\theta,\varphi) \\ \frac{\partial\Phi_z}{\partial\theta}(\theta,\varphi) & \frac{\partial\Phi_z}{\partial\varphi}(\theta,\varphi) \end{bmatrix}\right) = \det\left(\begin{bmatrix} -r\sin(\varphi)\sin(\theta) & r\cos(\varphi)\cos(\theta) \\ 0 & -r\sin(\varphi) \end{bmatrix}\right)$$

$$= r^2\sin^2(\varphi)\sin(\theta),$$

$$\frac{\partial(\Phi_y,\Phi_z)}{\partial(\theta,\varphi)} = \det\left(\begin{bmatrix} \frac{\partial\Phi_y}{\partial\theta}(\theta,\varphi) & \frac{\partial\Phi_y}{\partial\varphi}(\theta,\varphi) \\ \frac{\partial\Phi_z}{\partial\theta}(\theta,\varphi) & \frac{\partial\Phi_z}{\partial\varphi}(\theta,\varphi) \end{bmatrix}\right) = \det\left(\begin{bmatrix} r\sin(\varphi)\cos(\theta) & r\cos(\varphi)\sin(\theta) \\ 0 & -r\sin(\varphi) \end{bmatrix}\right)$$

$$= r^2\sin^2(\varphi)\cos(\theta).$$

Let's rewrite this a little bit more neatly:

$$\frac{\partial(\Phi_x,\Phi_y)}{\partial(\theta,\varphi)} = -r^2\cos(\varphi)\sin(\varphi),$$

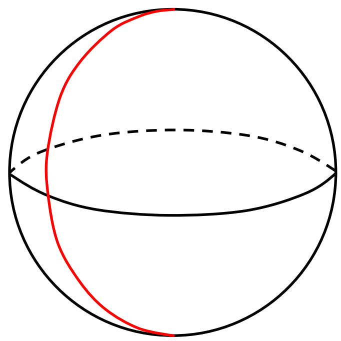
$$\frac{\partial(\Phi_x,\Phi_z)}{\partial(\theta,\varphi)} = r^2\sin^2(\varphi)\sin(\theta),$$

$$\frac{\partial(\Phi_y, \Phi_z)}{\partial(\theta, \varphi)} = r^2 \sin^2(\varphi) \cos(\theta).$$

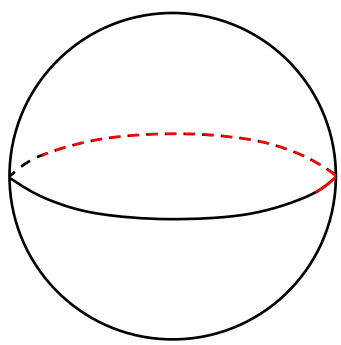
Notice that these don't ever vanish simultaneously on $\mathrm{dom}(\Phi) = (0, 2\pi) \times (0, \pi)$ and so Φ satisfies the 3rd condition of a surface parametrization. With this we have shown that Φ is a surface parametrization. Notice that Φ is surface parametrization that doesn't cover the whole sphere. If you look at the range of Φ (which you can do by graphing Φ) you'll notice that Φ covers the whole sphere except for a sliced portion of the sphere (the curve that is associated to $\theta = 0$ and $\varphi \in [0, 2\pi]$ under the image of Φ, shown in red in the image below). So the U and V in the definition of a surface parametrization (Definition 4.2.2) in this context are:

$$U = (0, 2\pi) \times (0, \pi)$$

$$V = \mathbb{R}^3 \backslash \{(r \sin(\varphi) \cos(\theta), r \sin(\varphi) \sin(\theta), r \cos(\varphi)) : \theta = 0, \varphi \in [0, 2\pi]\}.$$



Thus here the surface parametrization Φ only helps in proving that the sphere minus the red sliced portion in the image above is a smooth surface according to Definition 4.2.2. To show that the rest of the sphere is a smooth surface, you'll have to use at least one more surface parametrization, one choice being a rotated version of the spherical coordinates where the sliced-out region does not intersect the above sliced out region:

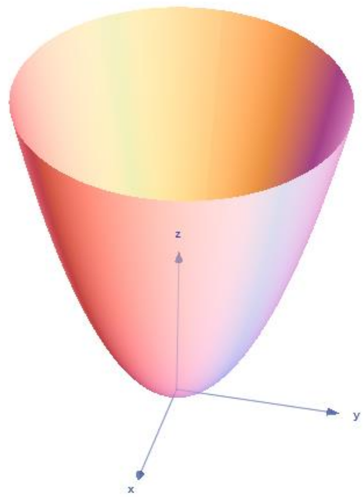


That way you will be able to cover the whole sphere with two surface parametrizations and thus prove that the sphere is a regular surface according to Definition 4.2.2. Spherical coordinates appear a lot in mathematics since they often make integration in $\mathbb{R}^3$ much easier.

Example 4.2.4: Let us take the function $\Phi : (0, \infty) \times (0, 2\pi)$ defined by:

$$\Phi(r, \theta) = (r \cos(\theta), r \sin(\theta), r^2)$$

I will leave the verification that this is indeed a surface parametrization to the reader. If you plot this surface parametrization you will see that this surface parametrization generates the paraboloid surface of revolution.



Thus, the surface parametrization Φ is a convenient way to parametrize paraboloid (minus a slice along the backside of the paraboloid unfortunately that is associated to the values $r \in [0, \infty)$ and $\theta = 0$). In fact, functions of the form $\Phi(r, \theta) = \big(r\cos(\theta), r\sin(\theta), f(r)\big)$ are good ways to parametrize any surface of revolution that you get by taking a function $y = f(x)$ and rotating it around the y-axis.

Example 4.2.5: Anything that is the graph of a function can be parametrized by a surface parametrization. For example, suppose that a surface S is given by the graph of the function $z = f(x, y)$ where the domain of f is an open set U and $f \in C^{\infty}[U]$. Then we can parametrize the surface S by the surface parametrization $\Phi : U \to \mathbb{R}^3$ given by:

$$\Phi(x, y) = \big(x, y, f(x, y)\big)$$

I will let the reader verify that this Φ satisfies the 3 properties of a surface parametrization. Notice that indeed if you plot this Φ you will get the surface S that is the graph of the function $z = f(x, y)$. In the case of the paraboloid this surface parametrization is given by $\Phi(x, y) = (x, y, x^2 + y^2)$ where $U = \mathbb{R}^2$. This surface parametrization is called the "**graph surface parametrization**" and it in fact proves one direction in the proof of the equivalence of Definitions 4.2.1 and 4.2.2 (to be discussed soon).

We mentioned above that if we already know that S is a smooth 2-dimensional surface and we just want to check that Φ is a surface parametrization of S, then it is sufficient to check only that Φ is a bijection and that it satisfies the first and third conditions of a surface parametrization. This is stated in the following theorem.

Theorem 4.2.6: *Let S be a smooth 2-dimensional surface sitting in $\mathbb{R}^3$ and let $\Phi : U \to V \cap S$ be a function as in the beginning of the statement of Definition 4.2.2 that satisfies properties 1 and 3 in the definition of a surface parametrization. Suppose also that Φ is a bijection between U and*

$V \cap S$. Then Φ and Φ^{-1} are continuous. In other words, Φ satisfies the 2nd condition in the definition of a surface parametrization and thus Φ is a surface parametrization.

The above theorem is really helpful when we deal with specific surfaces and their parametrizations because often times we already know that something is already a smooth surface because we were able to parametrize it in a different surface parametrization. An example of this is the case of the sphere and the spherical coordinates surface parametrization above (Example 4.2.3). I won't prove this theorem here as it will not be used anywhere else in this book and it not the main focus of our study of differential geometry [see future edition of this book though]. I do though think that it is important enough to include for the purpose of completion.

Now let us prove the equivalence of the definitions of a smooth surface given by Definitions 4.2.1 and 4.2.2. They are two ways of seeing the same thing and so let us prove that this is so. Before we prove this theorem, let us prove a lemma:

Lemma 4.2.7: *Let S be a smooth 2-dimensional surface sitting in $\mathbb{R}^3$. Suppose that $G : S \to \mathbb{R}^2$ is a continuous function and let U be an open set in $\mathbb{R}^2$. Then the preimage of U under G is equal to $S \cap V$ for some open set $V \subseteq \mathbb{R}^3$:*

$$G^{-1}[U] = S \cap V.$$

Proof: This is proved similarly to the standard theorem that says that the preimage of an open set under a continuous function of the form $F : \mathbb{R}^m \to \mathbb{R}^n$ is also an open set. However, things are a little bit trickier here since the domain of our function G is not a set of the form $\mathbb{R}^m$ but rather a subset of such a set (a smooth surface in fact). Nevertheless, the steps in this proof should resemble exactly those taken in the proof of the above-mentioned theorem. Take any point $p \in G^{-1}[U]$. Since U is open, there exists a small radius $r_p > 0$ such that the ball $B_{r_p}\big(G(p)\big)$ centered at $G(p)$ or radius r_p is contained in U:

$$B_{r_p}\big(G(p)\big) \subseteq U.$$

Since G is continuous, there exists a distance $R_p > 0$ such that for any $q \in S : \|q - p\| < R_p$, $\|G(q) - G(p)\| < r_p$. In other words:

$$G\left[B_{R_p}(p) \cap S\right] \subseteq B_{r_p}\big(G(p)\big).$$

The previous equation then implies that $G\left[B_{R_p}(p) \cap S\right] \subseteq U$ and thus $B_{R_p}(p) \cap S \subseteq G^{-1}[U]$. Since p can be any point in $G^{-1}[U]$ and every $B_{R_p}(p) \cap S \subseteq G^{-1}[U]$, we have that:

$$G^{-1}[U] = \bigcup_{p \in G^{-1}[U]} \left(B_{R_p}(p) \cap S\right) = S \cap \bigcup_{p \in G^{-1}[U]} B_{R_p}(p).$$

$\bigcup_{p \in G^{-1}[U]} B_{R_p}(p)$ is an open set because it the union of open sets. Thus if we set $V = \bigcup_{p \in G^{-1}[U]} B_{R_p}(p)$, we get that the above equation finally becomes:

$$G^{-1}[U] = S \cap V,$$

which is what we wanted to prove.

■

There are a few comments that I would like to make about the above theorem. First of all, nowhere in the above proof did we use the fact that S was a smooth surface. In fact, the above theorem holds equally well if S is replaced with any arbitrary subset of $\mathbb{R}^3$ and it can in fact be generalized to functions of the form $G : S \subseteq \mathbb{R}^m \to \mathbb{R}^n$. Also, as mentioned in the above proof there is a theorem in calculus that says that the preimage of any open set under a continuous function of the form $F : \mathbb{R}^m \to \mathbb{R}^n$ is also an open set. This theorem can be generalized to include the above theorem as a corollary, but such a thing requires the generalization of the notion of an open set to other spaces. In the case of the above theorem this generalization of the notion of an open set to the space S is called the "subspace topology" of S. This just one type of question that is explored in an important field of mathematics called to "topology." But to us however, the above lemma will help us prove the equivalence of Definitions 4.2.1 and 4.2.2.

Theorem 4.2.8: *Definitions 4.2.1 and 4.2.2 of smooth 2-dimensional surfaces sitting in* $\mathbb{R}^3$ *are equivalent.*

Proof: To prove that the two definitions of smooth surfaces are equivalent, we just have to show that S is a smooth surface according to Definition 4.2.1 if and only if S is a smooth surface according to Definition 4.2.2. One direction was actually already proven in Example 4.2.5, but let us repeat the argument here again. Suppose that S is a smooth surface according to Definition 4.2.1. We want to prove that S is a smooth surface according to Definition 4.2.2 (this is in fact the much easier direction to prove). Take any point $p \in S$ on the surface S. Since S is a smooth surface according to Definition 4.2.1, we can locally represent S is the graph of a function. Let us suppose without loss of generality that locally to p we can represent S as the graph of a function of the form $z = f(x, y)$ where $\text{dom}(f) = U$ is some open set (the cases when S is locally the graph of functions of the form $x = f(y, z)$ or $y = f(x, z)$ are handled similarly). Then we can form the surface parametrization $\Phi : U \to \mathbb{R}^3$ defined by:

$$\Phi(x, y) = \big(x, y, f(x, y)\big)$$

and notice that this is indeed a surface parametrization of the surface S located at the point p (I'll let the reader verify that). Since $p \in S$ was chosen arbitrarily on the surface S, we get that this shows that S is a smooth surface according to Definition 4.2.2.

Now let us prove the other direction. Suppose that S is a smooth surface according to Definition 4.2.2. We want to prove that S is a smooth surface according to Definition 4.2.1. This direction is in essence an application of the inverse function theorem. Take any point $p_0 = (x_0, y_0, z_0) \in S$ on the surface S. Since S is a smooth surface according to Definition 4.2.2, we know that there exists a surface parametrization Φ of S located at p_0. Let $(u_0, v_0) = \Phi^{-1}(x_0, y_0, z_0)$. Since Φ satisfies the 3$^{\text{rd}}$ condition of a surface parametrization, one of the following Jacobians is nonzero at (u_0, v_0):

$$\frac{\partial(\Phi_x,\Phi_y)}{\partial(u,v)}(u_0,v_0), \frac{\partial(\Phi_x,\Phi_z)}{\partial(u,v)}(u_0,v_0), \frac{\partial(\Phi_y,\Phi_z)}{\partial(u,v)}(u_0,v_0).$$

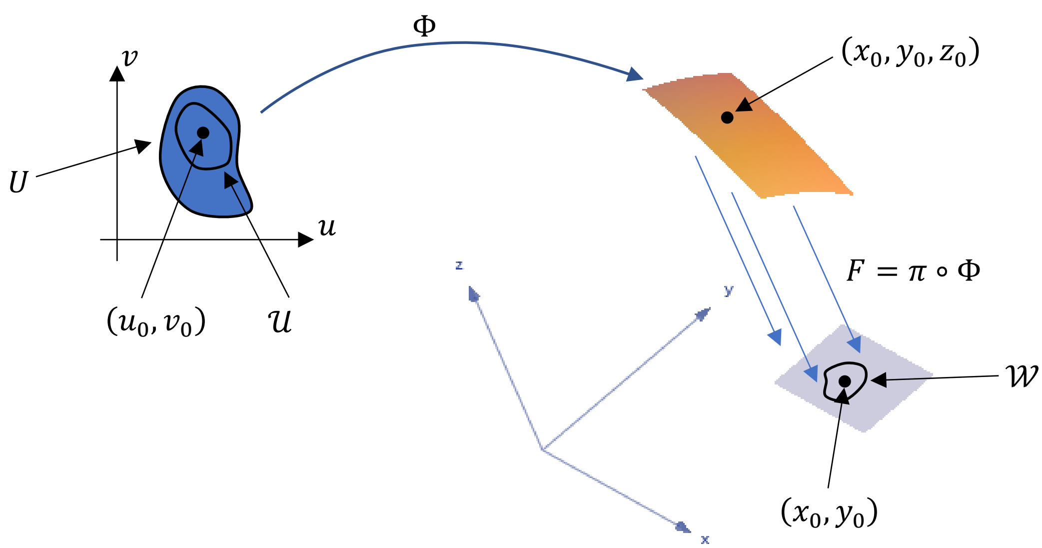
Let's suppose without loss of generality that the Jacobian $\frac{\partial(\Phi_x,\Phi_y)}{\partial(u,v)}(u_0,v_0) \neq 0$ (the cases when you have to choose one of the other two Jacobians in order to get a nonzero Jacobian are handled similarly). Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denote the projection map defined by $\pi(x,y,z) = (x,y)$. Now, let us consider the function $F = \pi \circ \Phi$. This F represents the projection of any point on the surface onto the x-y plane. Indeed, if we write out F explicitly we will get that:

$$F(u,v) = \pi\big(\Phi(u,v)\big) = \Big(\Phi_x(u,v), \Phi_y(u,v)\Big),$$

and thus F is the projection of the point $\Big(\Phi_x(u,v), \Phi_y(u,v), \Phi_z(u,v)\Big)$ on the surface onto the x-y plane. Notice that $(x_0,y_0) = F(u_0,v_0)$. Now, let us calculate the Jacobian of F at (u_0,v_0). We have that:

$$\det\big(DF(u_0,v_0)\big) = \det\left(\begin{bmatrix} \frac{\partial\Phi_x}{\partial u}(u,v) & \frac{\partial\Phi_x}{\partial v}(u,v) \\ \frac{\partial\Phi_y}{\partial u}(u,v) & \frac{\partial\Phi_y}{\partial v}(u,v) \end{bmatrix}\right) = \frac{\partial(\Phi_x,\Phi_y)}{\partial(u,v)}(u_0,v_0) \neq 0.$$

Thus the Jacobian of F is not zero at (u_0,v_0). So, by the inverse function theorem we know that there exists some small open set $\mathcal{U}$ containing the point (u_0,v_0) such that F has a continuous (and in fact infinitely differentiable) inverse on $\mathcal{U}$. The inverse $F^{-1}(x,y)$ exists on some small open set $\mathcal{W}$ centered at (x_0,y_0).



Now, by Lemma 4.2.7 we have that there exists an open set $\mathcal{V}$ such that $S \cap \mathcal{V} = \Phi^{-1}[U] = \Phi[U]$ (which notice contains $p_0 = (x_0,y_0,z_0)$). Now, I claim that $S \cap \mathcal{V}$ is the graph of the function C^∞ function:

$$f(x, y) = \Phi_z \circ F^{-1} = \Phi_z \circ (\pi \circ \Phi)^{-1}$$

over $(x, y) \in \mathcal{W}$. This can be seen by just following the arrows in the above picture. Indeed, for any $(x, y) \in W$ let $(u, v) = F^{-1}(x, y) \in \mathcal{U}$. Then we have that:

$$(x, y) = F(u, v) = \big(\Phi_x(u, v), \Phi_y(u, v)\big),$$

and thus:

$$\big(x, y, f(x, y)\big) = \big(\Phi_x(u, v), \Phi_y(u, v), (\Phi_z \circ (\pi \circ \Phi)^{-1})(x, y)\big)$$

$$= \big(\Phi_x(u, v), \Phi_y(u, v), \Phi_z(u, v)\big).$$

which is $S \cap \mathcal{V}$. The fact that every point in $S \cap \mathcal{V}$ is in the graph of $f(x, y)$ over $\mathcal{W}$ comes from the fact that Φ is a bijection between $\mathcal{U}$ and $S \cap \mathcal{V}$ and F is a bijection between $\mathcal{U}$ and $\mathcal{W}$. Thus, we have shown above that the piece $S \cap \mathcal{V}$ of the surface that contains p_0 where $\mathcal{V}$ is open is the graph of a C^∞ function over an open set $\mathcal{U}$. Since p_0 was a point chosen arbitrarily on S, we get that this shows that S is a smooth surface according to Definition 4.2.1.

■

With the above theorem we have shown that the definitions of a smooth surface given in Definitions 4.2.1 and 4.2.2 are the same thing. Definition 4.2.2 is usually the favorite among differential geometers, but both definitions turn out to be crucial in the study of smooth surfaces.

An extremely important concept in the theory of surfaces is the concept of the tangent plane to the surface at a point. Colloquially speaking, for a surface S and a point $p \in S$ on it, the tangent plane is the plane that just barely touches the surface and passes through the point p. Of course this is only an approximate qualitive description of a tangent plane since if for example the surface is a plane itself, then the tangent plane will coincide with the surface itself. So a more rigorous description of the tangent plane goes as follows.

Definition 4.2.9: *Let S be a smooth 2-dimensional surface sitting in $\mathbb{R}^3$ and let $p \in S$ be any point on this surface. Let Γ be the set of all C^∞ curves of the form $\gamma : [-1, 1] \to \mathbb{R}^3$ that lie on the surface S ("lying on the surface" means that for any $t \in [-1, 1]$, $\gamma(t) \in S$) and that pass through the point p at time $t = 0$: $p = \gamma(0)$. Let $T_p(s)$ be the set:*

$$T_p(s) = \{\gamma'(0) : \gamma \in \Gamma\}.$$

*It turns out (as we will prove) that $T_p(s)$ is a plane in $\mathbb{R}^3$ and it is called the **tangent plane** to S at the point p. Furthermore, if Φ is a surface parametrization of S at p and $(u_0, v_0) = \Phi^{-1}(p)$, then the plane $T_p(S)$ is spanned by the two vectors $\frac{\partial \Phi}{\partial u}(u_0, v_0)$ and $\frac{\partial \Phi}{\partial v}(u_0, v_0)$:*

$$T_p(S) = span\left\{\frac{\partial \Phi}{\partial u}(u_0, v_0), \frac{\partial \Phi}{\partial v}(u_0, v_0)\right\}.$$

So in essence in the above definition we're describing the tangent plane as the plane spanned by all vectors that are tangent to the surface which we obtained by using surface curves. Let's prove that every statement in the above definition is correct in the following theorem. Before we prove this theorem, we are going to need a lemma that has applications in a variety of places in differential geometry since it allows use to look at surface curves from the perspective of their inverse images under surface parametrizations.

Lemma 4.2.10: *Let S be a smooth 2-dimensional surface sitting in $\mathbb{R}^3$ and let $\Phi : U \to V \cap S$ be a surface parametrization on this surface where the sets U and V are open in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Suppose that $\gamma : [a,b] \to \mathbb{R}^3$ is a C^∞ curve that lies on the surface and is entirely in V (in other words, $\gamma(t)$ lies entirely in the image of Φ). Then the inverse image of this curve under the surface parametrization Φ:*

$$\big(u(t), v(t)\big) = \Phi^{-1}\big(\gamma(t)\big)$$

is also a C^∞ curve lying in the domain of Φ.

Proof: This is again just an application of the inverse function theorem. Let:

$$\gamma(t) = \big(x(t), y(t), z(t)\big)$$

be the explicit form of the curve $\gamma(t)$. In order to prove that $\big(u(t), v(t)\big)$ is a C^∞ curve we just have to show that at every point $t_0 \in (a,b)$, $\big(u(t), v(t)\big)$ is C^∞ at t_0 (in other words, we're going to look at one point in $[a,b]$ at a time).

Here we have that $\big(u(t_0), v(t_0)\big) = \Phi^{-1}\big(\gamma(t_0)\big)$. Since Φ satisfies the 3rd condition of a surface parametrization, one of the following Jacobians is nonzero at (u_0, v_0):

$$\frac{\partial\big(\Phi_x, \Phi_y\big)}{\partial(u,v)}\big(u(t_0), v(t_0)\big), \frac{\partial(\Phi_x, \Phi_z)}{\partial(u,v)}\big(u(t_0), v(t_0)\big), \frac{\partial\big(\Phi_y, \Phi_z\big)}{\partial(u,v)}\big(u(t_0), v(t_0)\big).$$

As before, let's suppose without loss of generality that the Jacobian $\frac{\partial(\Phi_x,\Phi_y)}{\partial(u,v)}\big(u(t_0), v(t_0)\big) \neq 0$. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denote the projection map defined by $\pi(x,y,z) = (x,y)$. Now, let us consider the function $F = \pi \circ \Phi$. We saw this kind of F in the proof of Theorem 4.2.6 and we proved there that this F has non-zero Jacobian. Notice that $\big(u(t_0), v(t_0)\big) = F\big(x(t_0), y(t_0)\big)$. So by the inverse function theorem there exist some small ball $B_r\big(x(t_0), y(t_0)\big)$ centered at $\big(x(t_0), y(t_0)\big)$ such that F restricted to this ball is bijective and its inverse is C^∞. Now, since the curve $\big(x(t), y(t)\big)$ in the x-y plane is continuous, there exists some open time interval $(\alpha, \beta) \subseteq [a,b]$ centered at t_0 such that the curve $\big(x(t), y(t)\big)$ is contained in the ball $B_r\big(x(t_0), y(t_0)\big)$ for $t \in (\alpha, \beta)$. Notice that we now have that:

$$\forall t \in (\alpha, \beta), \qquad \big(u(t), v(t)\big) = F\big(x(t), y(t)\big).$$

Since F and $\gamma(t)$ (and hence $x(t)$, $y(t)$, and $z(t)$) are C^∞ we get that the right side of the above equation is C^∞. Hence the curve $\big(u(t), v(t)\big)$ is C^∞ on all of $t \in (\alpha, \beta)$ and in particular at our time $t = t_0$. Since $t_0 \in (a, b)$ was chosen arbitrarily, this shows that $\big(u(t), v(t)\big)$ is C^∞ on all of (a, b).

■

The above lemma is a powerful because as mentioned before it allows us to look at surface curves locally in terms of surface parametrizations and do calculus on them. In explanation, locally we can represent surface curves $\gamma(t)$ in the form $\Phi\big(u(t), v(t)\big)$ where u, v, Φ are all C^∞. Now we prove the theorem.

Theorem 4.2.11: *Looking at the above definition, the statements that $T_p(S)$ is a plane in $\mathbb{R}^3$ and that $T_p(S)$ is spanned by $\frac{\partial\Phi}{\partial u}(u_0, v_0)$ and $\frac{\partial\Phi}{\partial v}(u_0, v_0)$ are correct (here I use the notation from the above definition (Definition 4.2.9)).*

Proof: The proof of this theorem is not too difficult. We want to prove that $T_p(S)$ is the plane spanned by $\frac{\partial\Phi}{\partial u}(u_0, v_0)$ and $\frac{\partial\Phi}{\partial v}(u_0, v_0)$:

$$T_p(S) = span\left\{\frac{\partial\Phi}{\partial u}(u_0, v_0), \frac{\partial\Phi}{\partial v}(u_0, v_0)\right\}.$$

Let's prove this by showing that every point of $T_p(S)$ is in the span of $\frac{\partial\Phi}{\partial u}(u_0, v_0)$ and $\frac{\partial\Phi}{\partial v}(u_0, v_0)$ and that every point in the span of $\frac{\partial\Phi}{\partial u}(u_0, v_0)$ and $\frac{\partial\Phi}{\partial v}(u_0, v_0)$ is in $T_p(S)$. Symbolically this is written as: "let us prove that $T_p(S) \subseteq span\left\{\frac{\partial\Phi}{\partial u}(u_0, v_0), \frac{\partial\Phi}{\partial v}(u_0, v_0)\right\}$ and that $T_p(S) \supseteq span\left\{\frac{\partial\Phi}{\partial u}(u_0, v_0), \frac{\partial\Phi}{\partial v}(u_0, v_0)\right\}$" (proving set equalities in this manner is a standard trick). Let's first prove that $T_p(S) \subseteq span\left\{\frac{\partial\Phi}{\partial u}(u_0, v_0), \frac{\partial\Phi}{\partial v}(u_0, v_0)\right\}$.

Take any $\gamma'(0) \in T_p(S)$. The vector $\gamma'(0)$ came from some $\gamma(t)$ in the set Γ. Let us take our surface parametrization $\Phi : U \to V \cap S$ where U and V are open sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. We need to look at $\gamma(t)$ from the perspective of how it behaves under the surface parametrization Φ. But we can only do that on the time interval when $\gamma(t)$ enters the range of Φ. So to do that we notice that by the continuity of $\gamma(t)$ and the fact that $\gamma(0) = p$ is in the interior of the open set V, there exists some small open interval (α, β) centered at 0 such that $\gamma(t)$ is inside of $V \cap S$ for all $t \in (\alpha, \beta)$. Great! That means that we can take the inverse image of $\gamma(t)$ under Φ over $t \in (\alpha, \beta)$. Let $\big(u(t), v(t)\big)$ be the preimage curve of $\gamma(t)$ under Φ:

$$\forall t \in (\alpha, \beta), \qquad \big(u(t), v(t)\big) = \Phi^{-1}\big(\gamma(t)\big)$$

that lies in the domain of Φ: U. This allows us to rewrite $\gamma(t)$ from the perspective of the surface parametrization Φ: over $t \in (\alpha, \beta)$ the curve $\gamma(t)$ is given by:

$$\gamma(t) = \Phi\big(u(t), v(t)\big).$$

And notice that $\big(u(t), v(t)\big)$ passes through (u_0, v_0) at time $t = 0$ since $\gamma(0) = p$ and:

$$(u_0, v_0) = \Phi^{-1}(p) = \Phi^{-1}\big(\gamma(0)\big) = \big(u(0), v(0)\big).$$

We also have that $\big(u(t), v(t)\big)$ is C^∞ by Lemma 4.2.10 above. Now with this we can finally describe the vector $\gamma'(0)$ in terms of the surface parametrization Φ:

$$\gamma'(0) = \frac{d}{dt}\Big(\Phi\big((u(t), v(t))\big)\Big)\Big|_{t=0} = \frac{\partial \Phi}{\partial u}(u_0, v_0) \cdot u'(0) + \frac{\partial \Phi}{\partial v}(u_0, v_0) \cdot v'(0).$$

This shows that $\gamma'(0) \in span\left\{\frac{\partial \Phi}{\partial u}(u_0, v_0), \frac{\partial \Phi}{\partial v}(u_0, v_0)\right\}$. So we have shown that any $\gamma'(0) \in T_p(S)$ is in the span of $\frac{\partial \Phi}{\partial u}(u_0, v_0)$ and $\frac{\partial \Phi}{\partial v}(u_0, v_0)$. Since $T_p(S) = \{\gamma'(0) : \gamma \in \Gamma\}$, this shows that $T_p(S) \subseteq span\left\{\frac{\partial \Phi}{\partial u}(u_0, v_0), \frac{\partial \Phi}{\partial v}(u_0, v_0)\right\}$.

Now let us prove the other direction $T_p(S) \supseteq span\left\{\frac{\partial \Phi}{\partial u}(u_0, v_0), \frac{\partial \Phi}{\partial v}(u_0, v_0)\right\}$. Take any vector $a\frac{\partial \Phi}{\partial u}(u_0, v_0) + b\frac{\partial \Phi}{\partial v}(u_0, v_0)$ in $span\left\{\frac{\partial \Phi}{\partial u}(u_0, v_0), \frac{\partial \Phi}{\partial v}(u_0, v_0)\right\}$ where $a, b \in \mathbb{R}$. To show that this vector is in $T_p(S)$, we just have to show that there is some surface curve $\gamma(t)$ that goes through p such that its first derivative at p is this vector. Let's construct such a curve $\gamma(t)$ as follows. Take the curve:

$$\big(u(t), v(t)\big) = (u_0 + at, v_0 + bt)$$

in the domain of Φ. This is in fact the line that passes through the point $(u_0, v_0) = \Phi^{-1}(p)$ with the velocity vector ("first derivative vector" that is) (a, b). Since $\big(u(t), v(t)\big)$ is continuous and $U = \mathrm{dom}(\Phi)$ is open, there exists some small time interval (α, β) centered at 0 such that $\big(u(t), v(t)\big)$ is contained in U. Now, notice that the curve given by:

$$\gamma(t) = \Phi\big(u(t), v(t)\big)$$

is in Γ since it lies on the surface S and at $t = 0$,

$$\gamma(0) = \Phi\big(u(0), v(0)\big) = \Phi(u_0, v_0) = p.$$

And notice that the derivative of this surface curve $\gamma(t)$ at $t = 0$ is our vector $a\frac{\partial \Phi}{\partial u}(u_0, v_0) + b\frac{\partial \Phi}{\partial v}(u_0, v_0)$:

$$\gamma'(0) = \frac{d}{dt}\big(\Phi(u(t), v(t)\big)\Big|_{t=0} = \frac{\partial \Phi}{\partial u}\big(u(0), v(0)\big) \cdot u'(0) + \frac{\partial \Phi}{\partial v}\big(u(0), v(0)\big) \cdot v'(0)$$

$$= \frac{\partial \Phi}{\partial u}(u_0, v_0) \cdot a + \frac{\partial \Phi}{\partial v}(u_0, v_0) \cdot b,$$

which is indeed our vector! Since $\gamma \in \Gamma$, this shows that $a\frac{\partial \Phi}{\partial u}(u_0, v_0) + b\frac{\partial \Phi}{\partial v}(u_0, v_0)$ is in $T_p(s)$. Now, since $a\frac{\partial \Phi}{\partial u}(u_0, v_0) + b\frac{\partial \Phi}{\partial v}(u_0, v_0)$ was an arbitrarily chosen vector in

$span\left\{\frac{\partial\Phi}{\partial u}(u_0,v_0), \frac{\partial\Phi}{\partial v}(u_0,v_0)\right\}$, this shows that $T_p(S) \supseteq span\left\{\frac{\partial\Phi}{\partial u}(u_0,v_0), \frac{\partial\Phi}{\partial v}(u_0,v_0)\right\}$. Having proved inclusion in both directions, we finally get that:

$$T_p(S) = span\left\{\frac{\partial\Phi}{\partial u}(u_0,v_0), \frac{\partial\Phi}{\partial v}(u_0,v_0)\right\}.$$

So indeed $T_p(S)$ is a plane and it is spanned by $\frac{\partial\Phi}{\partial u}(u_0,v_0)$ and $\frac{\partial\Phi}{\partial v}(u_0,v_0)$.

■

Now that we have defined the tangent plane to a surface, it's now very easy to define what it means for a vector to be perpendicular to a surface. Indeed, a vector is defined to be perpendicular to a surface if it is perpendicular to the tangent plane.

Definition 4.2.12: *Let S be a smooth 2-dimensional surface sitting in $\mathbb{R}^3$ and $p \in S$ be any point on it. Then N is said to be "**normal**" or "**perpendicular**" to the surface S at the point p if it is perpendicular to the tangent plane $T_p(S)$ to the surface at the point p.*

By definition of $T_p(S)$ we have that in order to check that N is perpendicular to S at a point $p \in S$, it suffices to show that N is perpendicular to $\gamma'(0)$ for every $\gamma \in \Gamma$ where Γ is the set of curves described in Definition 4.2.9. However, since $T_p(S)$ is spanned by $\frac{\partial\Phi}{\partial u}(u_0,v_0)$ and $\frac{\partial\Phi}{\partial v}(u_0,v_0)$ where Φ is a surface parametrization of S at p and $(u_0,v_0) = \Phi^{-1}(p)$, we see that in order to check that N is normal to S at p it also suffices to show that N is perpendicular to both $\frac{\partial\Phi}{\partial u}(u_0,v_0)$ and $\frac{\partial\Phi}{\partial v}(u_0,v_0)$. Notice that since $\frac{\partial\Phi}{\partial u}(u_0,v_0) \times \frac{\partial\Phi}{\partial v}(u_0,v_0)$ is always perpendicular $\frac{\partial\Phi}{\partial u}(u_0,v_0)$ and $\frac{\partial\Phi}{\partial v}(u_0,v_0)$, we get that $\frac{\partial\Phi}{\partial u}(u_0,v_0) \times \frac{\partial\Phi}{\partial v}(u_0,v_0)$ is a perpendicular vector to the surface S at the point p. Let's state this important result in a separate theorem.

Theorem 4.2.13: *Let S be a smooth 2-dimensional surface sitting in $\mathbb{R}^3$. Let Φ be a surface parametrization of S. Then for any $(u,v) \in \mathrm{dom}(\Phi)$, the vector:*

$$\frac{\partial\Phi}{\partial u}(u,v) \times \frac{\partial\Phi}{\partial v}(u,v)$$

is perpendicular to S (at the point $\Phi(u,v)$).

Proof: (The above paragraph).

An interesting application of the above theorem to surfaces that are represented as the graph of a function is the following: suppose that S is the graph of the function $z = f(x,y)$. What is a normal vector to the surface at $\left(x_0, y_0, f(x_0,y_0)\right)$? Well, in order to apply the formula in the above theorem for a normal vector to the surface, let us take the graph surface parametrization $\Phi(x,y) = \left(x, y, f(x,y)\right)$ that parametrizes the surface S. Then, by the above formula we get that:

$$\frac{\partial \Phi}{\partial x}(x_0, y_0) \times \frac{\partial \Phi}{\partial y}(x_0, y_0) = \begin{bmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(x_0, y_0) \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(x_0, y_0) \end{bmatrix} = \begin{bmatrix} -\frac{\partial f}{\partial x}(x_0, y_0) \\ -\frac{\partial f}{\partial y}(x_0, y_0) \\ 1 \end{bmatrix}$$

is a normal vector to the surface S at the point p. This formula for a vector that is perpendicular to the graph of a function is often taught in a calculus course.

Section 3: Level Set Representations of a Surface

In the previous section we discussed two ways to represent a surface: one way through graphs of functions and another way through surface parametrizations. There is however another way to represent a surface which we in fact already encountered in the previous chapter. Another way to represent a surface is by representing it as the level set of an infinitely differentiable function $g : \mathbb{R}^3 \to \mathbb{R}$:

$$S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = c\}$$

where $c \in \mathbb{R}$ is some fixed constant. Like in the last chapter, we require that ∇g never vanishes on S. We represented surfaces in this form in the previous chapter, but the natural question that one might ask here is: are such sets S smooth surfaces according to our definitions in the previous section? The answer is yes and the way to show that this is so is by applying the implicit function theorem.

Theorem 4.3.1: *Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a C^∞ function (meaning that $g \in C^\infty[\mathbb{R}^3]$). Let S be the level set of g:*

$$S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = c\}$$

where $c \in \mathbb{R}$ is some fixed constant. Suppose also that ∇g never vanishes on S. Then S is a smooth surface.

Proof: To show that S is a smooth surface we have to show that at each point $p \in S$ on the surface, S can either be locally represented as the image of a surface parametrization or the graph of a function. The graph of a function turns out to be easier in this case since this turns out to be a quick application of the implicit function theorem. Take any point $p = (x_0, y_0, z_0) \in S$ on the surface S. Let's show that locally to p, S is the graph of a function. By the condition imposed on ∇g, we know that $\nabla g(p) \neq 0$. So one of the partials $\frac{\partial g}{\partial x}(p), \frac{\partial g}{\partial y}(p), \frac{\partial g}{\partial z}(p)$ is not zero. Let's suppose without loss of generality that $\frac{\partial g}{\partial z}(p) \neq 0$ (the cases when this partial is zero and you have to choose one of the other two partials in order to get a nonzero partial of g are handled similarly). Then by the implicit function theorem we can represent the level set $\{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = c\}$, or the surface S, as the graph of a C^∞ function of the form $z = f(x, y)$ near p. More precisely, there exist some open set U containing (x_0, y_0) and an open set V containing p such that the sset $\{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = c\} \cap V = S \cap V$ is the graph of the function $f(x, y)$ over $(x, y) \in U$. Basically what we've done here is locally to p we used the implicit function theorem to “solve” for z in the equation $g(x, y, z) = 0$ and got $z = f(x, y)$. Since $p \in S$ on the surface S was chosen arbitrarily, according to Definition 4.2.1 this shows that S is a smooth surface.

∎

It turns out that the converse of Theorem 4.3.1 also holds: every smooth surface can locally be represented as the level set of a function. This merely follows from the fact that since a smooth surface can locally be represented as the graph of a function, say $z = f(x, y)$, then we can set $g(x, y, z) = z - f(x, y)$ and we will get that S is locally the level set of the function $g(x, y, z)$. With this we get the following theorem:

Theorem 4.3.2: *Suppose that S is a smooth 2-dimensional surface sitting in $\mathbb{R}^3$. Then locally to any point $p \in S$, S can be represented as the level set of a C^∞ function. In other words, for any point $p \in S$ there exists an open set $V \subseteq \mathbb{R}^3$ that contains p and $V \cap S$ is the level set of a C^∞ function $g(x, y, z)$ such that the partials $\partial g/\partial x$, $\partial g/\partial y$, and $\partial g/\partial z$ never vanish simultaneously on $V \cap S$ (in other words, ∇g is never zero on $V \cap S$).*

One of the convenient properties of level set representations of surfaces is that they are easy to use to obtain a formula for the normal vector to the surface at any point on the surface. Indeed, in the level set representation $S = \{(x, y, z) \in S : g(x, y, z) = c\}$ where ∇g never vanishes on S, a vector that is perpendicular to the surface S at any point $(x, y, z) \in S$ is $\nabla g(x, y, z)$. This is a famous fact from multivariable calculus, but let’s review how this is proven.

Take any point $p = (x_0, y_0, z_0) \in S$ on the surface and take any C^∞ curve $\gamma : [-1, 1] \to \mathbb{R}^3$ that lies on the surface S and that passes through p at time $t = 0$. Then, since this curve lies on the surface S we have that:

$$g\big(\gamma(t)\big) = g\big(x(t), y(t), z(t)\big) \equiv c$$

on $t \in [-1, 1]$. In particular, this means that the derivative of the left-hand side is zero at $t = 0$:

$$\frac{d}{dt}\Big(g\big(\gamma(t)\big)\Big)\Big|_{t=0} = \nabla g(x_0, y_0, z_0) \cdot \gamma'(0) = 0.$$

So $\nabla g(x_0, y_0, z_0) = \nabla g(p)$ is always perpendicular to $\gamma'(0)$. Since $\gamma(t)$ was an arbitrary surface curve that passes through the point p, according to Definition 4.2.12 this shows that $\nabla g(p)$ is perpendicular to the surface S at the point $p \in S$. Thus, in order to obtain an equation for a normal vector to a surface that is the level set of a function g, you can always take ∇g to be your normal vector. A magic rule!

Let us look at some examples of surfaces that can be represented as the level set of a function.

Example 4.3.3: Take the function $g(x, y, z) = x^2 + y^2 + z^2$. Let us take the level set of g:

$$S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = x^2 + y^2 + z^2 = r\}.$$

where $r > 0$ is some fixed positive constant. What is this surface S? It is the set of points in $\mathbb{R}^3$ that are located a distance of r from the origin. In other words, S is the sphere of radius r. Notice that g satisfies the property that ∇g never vanishes on S since $0 \notin S$. Notice also that at every point $(x_0, y_0, z_0) \in S$ on this surface, a normal vector to the surface at this point is given by:

$$\nabla g(x_0, y_0, z_0) = (2x_0, 2y_0, 2z_0) = 2(x_0, y_0, z_0).$$

Thus, the normal direction to the sphere at a point $(x_0, y_0, z_0) \in S$ is always the direction that points radially out in the same direction as (x_0, y_0, z_0). A geometrically clear fact for the sphere.

Example 4.3.4: Take the function $g(x, y, z) = 4x - 2y + z$. Let us take the level set of g:

$$S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 4x - 2y + z = 3\}.$$

What is this surface S? In this case we can solve for z in the level set equation $4x - 2y + z = 3$ to get that this surface S is in fact the plane generated by the graph of:

$$z = 3 - 4x + 2y.$$

It isn't always the case that we can solve for z in the equation $g(x, y, z) = c$, but in this case we do have that nice property. Notice that a normal vector to the surface at any point $(x_0, y_0, z_0) \in S$ is given by:

$$\nabla g(x_0, y_0, z_0) = (4, -2, 1).$$

Thus the normal direction to the surface S always points in the same line: the line spanned by $(4, -2, 1)$. This should be obvious since we are dealing with a plane.

We talked about the fact that if a surface S is represented as the graph of a function $z = f(x, y)$, then it can be represented as the level set of a function. Just set $g(x, y, z) = z - f(x, y)$ and you will get that our surface S is the zero level set of the function g:

$$S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}.$$

Now, if we want to find an equation for a normal vector to the surface S at some point $\big(x_0, y_0, f(x_0, y_0)\big)$ on it, then by the above discussion such a normal vector is given by:

$$\nabla g\big(x_0, y_0, f(x_0, y_0)\big) = \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right)$$

The same equation that we got for a normal vector to a surface that is the graph of a function at the end of the previous section!

Having defined smooth surfaces and presented many of its representations, the next step is to define their curvature. Indeed, from experience we know that surfaces have the ability to turn and curve in many directions, but it isn't immediately clear how one should go about defining such curvature mathematically. The Gaussian and mean curvatures are one answer to this problem and we discuss them in the next section.

Section 4: Curvatures of a Surface

The curvatures of surfaces are a most interesting topic and it is pretty much, I believe, the central topic that differential geometry studies. The curvatures of a surface describe how a surface fundamentally curves in all directions. Notice that I keep saying "curvatures" instead of just "curvature." Indeed, there is no one way to describe the curvature of a surface and as we will see the Gaussian and mean curvatures of a surface together fundamentally describe the way a surface curves in every tangent direction at any point.

Let's start by supposing that we are given a surface S and the task of assigning a number to each point of the surface that describes how the surface fundamentally "curves" at that point. One number however does not seem to be enough to do the job because such a number has to describe how the surface curves in every tangent direction. Maybe two numbers are enough?

For a long time people didn't have a good idea on how to define the curvature of surfaces. People did however know for a long time how to define the curvature of a curve in 3-dimensional space. Indeed, if you want to find out how much a curve is curving just analyze its second derivative. If you want to find out how much a curve is curving into a certain direction, then you just have to analyze the curve's second derivative's component in that unit direction. So let's start with that and see where that leads us.

Definition 4.4.1: *Suppose that $\gamma : [t_0, t_1] \to \mathbb{R}^3$ is a $C^\infty[t_0, t_1]$ curve such that its derivative is always a unit vector: $\|\gamma'(t)\| \equiv 1$. Such curves where $\|\gamma'(t)\| \equiv 1$ are often called "**unit speed curves**" since their speed $\|\gamma'(t)\|$ is constantly one. Let:*

$$\gamma(t) = \big(u(t), v(t), w(t)\big)$$

be the explicit form of $\gamma(t)$. Let $\alpha = (\alpha_x, \alpha_y, \alpha_z)$ be a unit vector in $\mathbb{R}^3$:

$$\|\alpha\| = \sqrt{\alpha_x^2 + \alpha_y^2 + \alpha_z^2} = 1.$$

*Then the **component of curvature** of the curve $\gamma(t)$ in the unit direction α at time $t = t_0$ is defined as:*

$$k_{\gamma(t)}(\alpha)\Big|_{t=t_0} = \gamma''(t_0) \cdot \alpha = u''(t_0)\alpha_x + v''(t_0)\alpha_y + w''(t_0)\alpha_z.$$

Basically, what the component of curvature of a curve describes is how much a curve is "curving in" on a certain unit direction. And the reason behind why we require that the curve is unit speed comes from geometric considerations. If you for example reparametrize the curve to go two times faster, then the component of curvature of γ in the direction α would become four times bigger. So in order to make the component of curvature of a curve a geometric property (meaning independent of parametrization), mathematicians had to agree on a convention and so they decided to use unit speed parametrizations of curves to define them. Here we're just going to say that components of curvature are defined only for unit speed curves.

It's great that we can define curvature of curves, but how does that help us in our quest to find out how we should define curvature of surfaces. Since we looked at the second derivatives of curves in order to define their curvatures, maybe in order to define the curvature of a surface we have to look at its second derivative. That is in fact exactly what we're going to do, but things aren't going to be as easy as in the case of curves. The idea behind how you define the curvature of a surface goes along the following thoughts: "I don't exactly know how to describe the curvature of a surface S at a point $p \in S$, but what I can do is describe how much the curves that lie on the surface S and that go through the point p curve into the surface at the point p. Maybe using those quantities I can describe the curvature of the surface S at that point."

Basically the idea behind how one can describe the curvature of a surface at a point p is to look at how much curves that go through the point p and that lie on the surface "curve into the surface at that point." More precisely we want to look at the components of curvature of the curves that lie on the surface and that go through the point p in the unit normal direction to the surface. By looking at all such possible components of curvature, we will then be able to fundamentally describe how the surface curves at that point. So let's do this.[23]

Before we get to the rigorous definition of a surface curvatures, let us loosely discuss the curving nature of surfaces for the purpose of getting an intuitive idea behind why the curvatures of surfaces are defined the way they are. Let's suppose that we have a smooth surface S and a point $p \in S$ where we want to describe the curvature of the surface S at. To make calculations easier, let's translate and rotate the surface S so that the tangent plane to the surface at p becomes perpendicular to the z-axis and p becomes right above the point $(0,0)$ on the x-y plane. Let's call the new rotated surface S' and the point that p rotates to p'. So we get that S' is a new smooth surface for which the tangent plane at p' is perpendicular to the z-axis and the projection of p' onto the x-y plane is $(0,0)$. Now, since S' is a smooth surface we can represent S' locally to p' as the graph of a function $z = f(x, y)$ (we can't represent it as the graph of $y = f(x, z)$ or $x = f(y, z)$ since if you geometrically visualize this, the tangent plane to S at p' is perpendicular to the z-axis). Notice that our point p' is equal to:

$$p' = \big(0,0, f(0,0)\big).$$

Let us take any unit speed curve $\gamma \in C^\infty[-1, 1]$ that lies on the surface S' and that passes through p' at time $t = 0$:

$$\gamma(0) = p' = \big(0,0, f(0,0)\big).$$

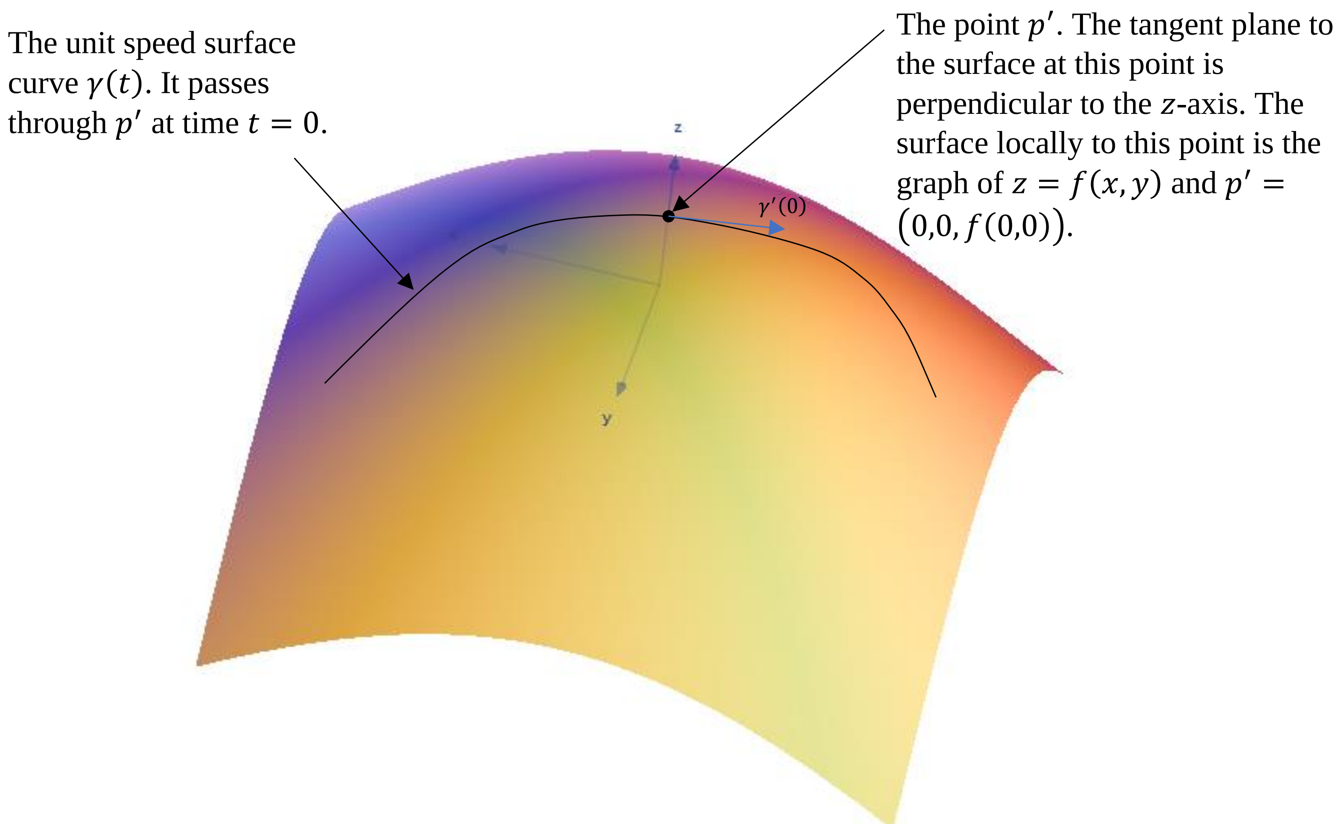
Since I like explicit forms, let:

$$\gamma(t) = \big(u(t), v(t), w(t)\big)$$

be the explicit form of $\gamma(t)$. With this we can rewrite the previous equation as:

[23] By the way, all of the following is just mathematical discussion that will lead into the definition of the curvatures of a surface. Its purpose is just to give the idea behind why the curvatures of a surface are defined the way they are and so if you would rather not read such a discussion you can skip right to the definitions without loss of continuity.

$$Equation\ 4.4.2: \qquad (u(0), v(0), w(0)) = p' = (0,0, f(0,0)).$$



Because S' locally to p' can be represented as the graph of the function $z = f(x, y)$, we get by the continuity of $\gamma(t)$ that there exists some small open time interval (α, β) centered at 0 such that $\gamma(t)$ is close enough to p' and so for any $t \in (\alpha, \beta)$ we can express $w(t)$ as:

$$w(t) = f(u(t), v(t)).$$

In other words, we get that:

$$\forall t \in (\alpha, \beta), \qquad \gamma(t) = (u(t), v(t), f(u(t), v(t))).$$

Great! This allows us to do some explicit calculus on γ and S' in terms of $f(x, y)$. To see how much the curve $\gamma(t)$ is curving "into the surface" at the point p', let us calculate its component of curvature at p' in the unit normal direction to the surface at p'. Since the tangent plane to S' at p' is perpendicular to the z-axis, the unit vector $(0,0,1)$ is perpendicular to the surface S' at p'. So let's calculate the component of curvature of $\gamma(t)$ at p' in the unit direction $(0,0,1)$ (the other option for the unit normal direction to the surface S' at p' is $(0,0,-1)$ and it works just as fine except that all of the signs are flipped. I just decided to use $(0,0,1)$). $\gamma(t)$ passes through p' at $t = 0$ and so in more refined language we should say: "let's calculate the component of curvature of $\gamma(t)$ in the unit direction $(0,0,1)$ at time $t = 0$." Doing this gives us that:

$$k_{\gamma(t)}(0,0,1)\big|_{t=0} = \gamma''(0)\cdot(0,0,1) = w''(0).$$

For $t \in (\alpha, \beta)$, $w(t) = f\big(u(t), v(t)\big)$ and so:

$$k_{\gamma(t)}(0,0,1)\big|_{t=0} = w''(0) = \frac{d^2}{dt^2}\Big(f\big(u(t), v(t)\big)\Big)\Big|_{t=0}$$

$$= \frac{\partial^2 f}{\partial x^2}\big(u(0), v(0)\big)\big(u'(0)\big)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\big(u(0), v(0)\big)u'(0)v'(0) + \frac{\partial^2 f}{\partial x^2}\big(u(0), v(0)\big)\big(v'(0)\big)^2$$

$$+\frac{\partial f}{\partial x}\big(u(0), v(0)\big)u''(0) + \frac{\partial f}{\partial y}\big(u(0), v(0)\big)v''(0).$$

By Equation 4.4.2 we have that this is equal to:

$$k_{\gamma(t)}(0,0,1)\big|_{t=0} = \frac{\partial^2 f}{\partial x^2}(0,0)\big(u'(0)\big)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0,0)u'(0)v'(0) + \frac{\partial^2 f}{\partial x^2}(0,0)\big(v'(0)\big)^2$$

$$+\frac{\partial f}{\partial x}(0,0)u''(0) + \frac{\partial f}{\partial y}(0,0)v''(0).$$

Since the tangential plane to S' at p' is perpendicular to the z-axis, we have that $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$. So the above equation becomes:

$$k_{\gamma(t)}(0,0,1)\big|_{t=0} = \frac{\partial^2 f}{\partial x^2}(0,0)\big(u'(0)\big)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0,0)u'(0)v'(0) + \frac{\partial^2 f}{\partial x^2}(0,0)\big(v'(0)\big)^2.$$

Wow, so basically we get that the component of curvature of $\gamma(t)$ at p' in the unit normal direction to the surface at p' is given by the quadratic term of the Taylor series of $f(x,y)$. So to analyze the curvature of the surface S' at p' we do start, as promised, by looking at the second derivatives of the surface. Quadratic terms of Taylor series can be rewritten in quadratic matrix forms. Indeed, notice that the above quantity can be rewritten as:

$$k_{\gamma(t)}(0,0,1)\big|_{t=0} = [u'(0) \quad v'(0)]\cdot\begin{bmatrix}\frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0)\\ \frac{\partial^2 f}{\partial x \partial y}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0)\end{bmatrix}\begin{bmatrix}u'(0)\\ v'(0)\end{bmatrix}$$

$$= [u'(0) \quad v'(0)]\cdot\mathcal{H}\begin{bmatrix}u'(0)\\ v'(0)\end{bmatrix} = \left\langle\begin{bmatrix}u'(0)\\ v'(0)\end{bmatrix}, \mathcal{H}\begin{bmatrix}u'(0)\\ v'(0)\end{bmatrix}\right\rangle$$

where $\mathcal{H}$ is the Hessian matrix of f at $(0,0)$ and $\langle \quad \rangle$ denotes the standard inner product in $\mathbb{R}^2$. Converting quadratic terms of Taylor series into quadratic matrix forms is a powerful technique because it allows us to use our powerful theorems from linear algebra to study the quantity in question, especially since the matrix involved is almost always symmetric. Notice that the vector $\begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix}$ that we're plugging into the quadratic matrix form above is a unit vector since:

$$1 = \|\gamma'(0)\| = \sqrt{\big(u'(0)\big)^2 + \big(v'(0)\big)^2 + \big(w'(0)\big)^2}$$

$$= \sqrt{\big(u'(0)\big)^2 + \big(v'(0)\big)^2 + \frac{\partial f}{\partial x}\big(u(0), v(0)\big)u'(0) + \frac{\partial f}{\partial y}\big(u(0), v(0)\big)v'(0)}$$

$$= \sqrt{\big(u'(0)\big)^2 + \big(v'(0)\big)^2 + \frac{\partial f}{\partial x}(0,0)u'(0) + \frac{\partial f}{\partial y}(0,0)v'(0)} = \sqrt{\big(u'(0)\big)^2 + \big(v'(0)\big)^2}$$

$$= \left\| \begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix} \right\|.$$

And the unit vector $\begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix}$ can in fact potentially be any unit vector in $\mathbb{R}^2$ since the curve $\gamma(t)$ can approach the point p' from any direction. So let us replace the vector $\begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix}$ in the above quadratic form by $\begin{bmatrix} u \\ v \end{bmatrix}$ where $\begin{bmatrix} u \\ v \end{bmatrix}$ can denote any unit vector in $\mathbb{R}^2$. Then the above component of curvature function of any curve $\gamma(t)$ that approaches the point p' with unit tangent vector (u, v) becomes:

$$k(u, v) = \langle \begin{bmatrix} u \\ v \end{bmatrix}, \mathcal{H} \begin{bmatrix} u \\ v \end{bmatrix} \rangle.$$

This equation describes how much the curves that lie on the surface S' and that pass through the point p' with unit tangent vector (u, v) curve into the surface at the point p'. So the above formula fundamentally describes how much the surface "curves" in on itself while moving in the direction (u, v). One of the nice things that we have in the above quadratic form is that the matrix $\mathcal{H}$ is symmetric. Thus we can apply the spectral theorem to get that there exists an orthonormal basis of eigenvectors of the matrix $\mathcal{H}$ such that in this basis the matrix $\mathcal{H}$ is diagonal. Rewriting the above equation in this new orthonormal basis of eigenvectors, we get that the above equation becomes:

$$k(U, V) = \langle \begin{bmatrix} U \\ V \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} \rangle$$

where λ_1 and λ_2 are the eigenvalues of the matrix $\mathcal{H}$ and (U, V) is a unit vector in this new basis. Let's rewrite the above expression in its explicit form. Multiplying out the above equation gives:

$$k(U, V) = \lambda_1 U^2 + \lambda_2 V^2.$$

So by diagonalization we have in fact gotten rid of the mixed term in the quadratic form. This procedure can in fact can be interpreted as that we rotated the function $f(x, y)$ so that the mixed partials term in its Taylor series went away. The fact that this can always be done is remarkable and it is given to us by the spectral theorem (it works in higher dimensions as well!). One of the above eigenvalues is smaller than or equal to the other. Let's suppose that $\lambda_1 \leq \lambda_2$ (the case when $\lambda_2 > \lambda_1$ is analyzed similarly). It turns out that λ_1 and λ_2 represent the minimum and maximum curvatures of the surface in all unit direction (U, V). They are attained by the eigenvector directions (1,0) and (0,1):

$$k(1,0) = \lambda_1,$$

$$k(0,1) = \lambda_2,$$

and for any unit vector (U, V) we have that $\|(U, V)\|^2 = U^2 + V^2 = 1$ and thus:[24]

$$k(U, V) = \lambda_1 U^2 + \lambda_2 V^2 \geq \lambda_1 (U^2 + V^2) = \lambda_1$$

$$k(U, V) = \lambda_1 U^2 + \lambda_2 V^2 \leq \lambda_2 (U^2 + V^2) = \lambda_2$$

So the eigenvalues λ_1 and λ_2 indeed represent the minimum and the maximum of the surface curvatures in all of the unit directions (U, V). As a result, these curvature values λ_1 are λ_2 describe the range of all of the curvatures of the surface in all directions at the point p'. For this reason, these curvatures values λ_1 and λ_2 are called the "principal curvatures" and their associated directions (the eigenvectors) are called the "principal directions." Notice that these principal directions are perpendicular to each other since the eigenvectors form an orthogonal basis of $\mathbb{R}^2$. Gauss looking at these principal curvatures decided to define his "Gaussian curvature" of the surface S' at the point p' as the product of these two principal curvatures:

$$K = \lambda_1 \cdot \lambda_2 = \det\left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\right) = \det(\mathcal{H}).$$

(det$\left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\right) = \det(\mathcal{H})$ because the matrix $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ is obtained from $\mathcal{H}$ by a change of basis and determinants are invariant under change of basis). And the "mean curvature" of the surface S' at p' is defined as the average of the two principal curvatures:

$$H = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2} trace\left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\right) = \frac{1}{2} trace(\mathcal{H}).$$

($trace\left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\right) = trace(\mathcal{H})$ because just like determinants traces of matrices are also invariant under change of basis). Since we translated and rotated the surface S to obtain S' and the above "component of curvature mathematics" involving surface curves is invariant under translations and rotations (this comes from the fact that the standard inner product in $\mathbb{R}^3$ is invariant under these transformations), the above values K and H are taken to be the definition of the Gaussian and mean curvatures of the surface S at the original point p respectively.

Thus we arrive at a principle: at any point p of a surface S if you look at the component of curvature of the curves that lie on the surface S and that pass through the point p in the normal

[24] The reason why the Pythagorean formula holds for (U, V): $\|(U, V)\|^2 = U^2 + V^2$ is because we are working in an orthonormal basis (the eigenvectors).

direction to the surface at p, you will get that there are two tangential directions that they can pass through p with that give you the minimum and the maximum components of curvature. And these two directions are perpendicular to each other. These minimum and maximum components of curvature are called the "principal directions" and their product and average are defined as the Gaussian and mean curvatures of the surface at the point p.[25]

Rotating a surface so that the z-axis is perpendicular to the surface at some point each time in order to calculate the surface curvatures there is really difficult and a lot of work. Fortunately there is a much easier way to calculate surface curvatures and it is done through what's called the "Gauss map." Let's see what this wonderful map is.

Suppose that we have a smooth surface S and a point $p \in S$ on it where we want to calculate the surface curvatures of S at. Since S is a smooth surface, there exists a surface parametrization $\Phi : U \to V \cap S$ of S at p. Since $\Phi(u, v)$ is a surface parametrization of S at p, there exists some (u_0, v_0) in the domain of Φ such that:

$$p = \Phi(u_0, v_0).$$

We know by definition that the domain U of this surface parametrization is open and so there exists some small open ball $B_r(u_0, v_0)$ around (u_0, v_0) such that $B_r(u_0, v_0) \subseteq U = \text{dom}(\Phi)$. Now the Gauss map is a function $N : U \to \mathbb{R}^3$ that does the following: at each point $(u, v) \in E$, $N(u, v)$ is a unit vector that is perpendicular to the surface at the point $\Phi(u, v)$ on the surface. One way to construct such a Gauss map is to set:

$$N(u, v) = \frac{\dfrac{\partial \Phi}{\partial u}(u, v) \times \dfrac{\partial \Phi}{\partial v}(u, v)}{\left\| \dfrac{\partial \Phi}{\partial u}(u, v) \times \dfrac{\partial \Phi}{\partial v}(u, v) \right\|}.$$

Obviously this is a unit vector (notice that the denominator is never zero by condition 3 of a surface parametrization). And it is indeed perpendicular to the surface S at the point $\Phi(u, v)$ for every $(u, v) \in U$ by Theorem 4.2.12. Additionally, notice that this Gauss map is infinitely differentiable since $\Phi \in C^\infty[U]$. Now, let us use the Gauss map to calculate the curvature of the surface S at the point $p = \Phi(u_0, v_0)$.

The idea behind how one calculates the surface curvatures of S at $p = \Phi(u_0, v_0)$ is exactly the same as before. We need to express the components of curvature of the surface curves that pass through the point $p = \Phi(u_0, v_0)$ in the normal direction $N(u, v)$ as a quadratic matrix form. We will then analyze the spectrum of the matrix sitting in this quadratic matrix form to get our principal curvatures and ultimately the surface curvatures of S. Take any unit speed C^∞ curve $\gamma : [-1, 1] \to \mathbb{R}^3$ that lies on the surface S and that passes through p at time $t = 0$. Like before, we want to look at the surface curve $\gamma(t)$ from the perspective of the surface parametrization Φ. To do that we notice that by the continuity of $\gamma(t)$ and the fact that $\gamma(0) = p$ is in the interior of the open set V, there exists some small open interval (α, β) centered at 0 such that $\gamma(t)$ is inside of

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[25] There is a technicality when these minimum and the maximum components of curvature are equal. As we will discuss below such points are called "umbilical points" and the statement above isn't quite true.

$V \cap S$ for all $t \in (\alpha, \beta)$. Great! That means that we can take the inverse image of $\gamma(t)$ under Φ over $t \in (\alpha, \beta)$. Let $\big(u(t), v(t)\big)$ be the preimage curve of $\gamma(t)$ under Φ:

$$\forall t \in (\alpha, \beta), \qquad \big(u(t), v(t)\big) = \Phi^{-1}\big(\gamma(t)\big)$$

that lies in the domain of Φ: U. This allows us to rewrite $\gamma(t)$ from the perspective of the surface parametrization Φ: over $t \in (\alpha, \beta)$ the curve $\gamma(t)$ is given by:

$$\gamma(t) = \Phi\big(u(t), v(t)\big).$$

And notice that at time $t = 0$ the inverse image curve $\big(u(t), v(t)\big)$ passes through (u_0, v_0):

$$\big(u(0), v(0)\big) = \Phi^{-1}\big(\gamma(0)\big) = \Phi^{-1}(p) = (u_0, v_0).$$

These equations will allow us to calculate the component of curvature of $\gamma(t)$ in the unit normal direction to the surface in terms of the surface parametrization Φ. We have that the component of curvature of $\gamma(t)$ in the unit normal direction $N(u_0, v_0)$ to the surface at the point p (or more precisely: at $t = 0$) is given by (remember, here $\cdot$ is the vector dot product when it stands between two vectors):

$$k_{\gamma(t)}\big(N(u_0, v_0)\big)\Big|_{t=0} = \frac{d^2}{dt^2}\big(\gamma(t)\big)\bigg|_{t=0} \cdot N(u_0, v_0) = \frac{d^2}{dt^2}\Big(\Phi\big(u(t), v(t)\big)\Big)\bigg|_{t=0} \cdot N(u_0, v_0)$$

$$\left(\frac{\partial^2\Phi}{\partial u^2}(u_0, v_0)\big(u'(0)\big)^2 + 2\frac{\partial^2\Phi}{\partial u \partial v}(u_0, v_0)\big(u'(0)\big)\big(v'(0)\big) + \frac{\partial^2\Phi}{\partial v^2}(u_0, v_0)\big(v'(0)\big)^2 \right.$$
$$\left. + \frac{\partial\Phi}{\partial u}(u_0, v_0)\big(u''(0)\big) + \frac{\partial\Phi}{\partial v}(u_0, v_0)\big(v''(0)\big)\right) \cdot N(u_0, v_0).$$

Using the distributive property of the dot product $\cdot$ we get that (I am going to stop writing the arguments of the partials of Φ and N here; they are being evaluated at (u_0, v_0)):

$$k_{\gamma(t)}(N)\Big|_{t=0} = \big(u'(0)\big)^2 \frac{\partial^2\Phi}{\partial u^2} \cdot N + 2\big(u'(0)\big)\big(v'(t_0)\big)\frac{\partial^2\Phi}{\partial u \partial v} \cdot N + \big(v'(0)\big)^2 \frac{\partial^2\Phi}{\partial v^2} \cdot N$$
$$+ u''(0)\frac{\partial\Phi}{\partial u} \cdot N + v''(0)\frac{\partial\Phi}{\partial v} \cdot N.$$

Since $N = \frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v} \Big/ \left\|\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v}\right\|$, we have that $N \perp \frac{\partial\Phi}{\partial u}$ and $N \perp \frac{\partial\Phi}{\partial v}$ and so two of the above terms go away to give that:

$$k_{\gamma(t)}(N)\Big|_{t=0} = \big(u'(0)\big)^2 \frac{\partial^2\Phi}{\partial u^2} \cdot N + 2\big(u'(0)\big)\big(v'(0)\big)\frac{\partial^2\Phi}{\partial u \partial v} \cdot N + \big(v'(0)\big)^2 \frac{\partial^2\Phi}{\partial v^2} \cdot N.$$

Awesome! Let's get rid of the second partials of Φ in the above equation. Let's do this by "differential integrating" the terms $\frac{\partial^2\Phi}{\partial u^2} \cdot N, \frac{\partial^2\Phi}{\partial u \partial v} \cdot N, \frac{\partial^2\Phi}{\partial v^2} \cdot N$ by parts. We have that (here I will include the arguments of the partials of Φ and N for clarity):

$$\frac{\partial^2 \Phi}{\partial u^2}(u_0, v_0) \cdot N(u_0, v_0) \\ = \frac{d}{dt}\left(\frac{\partial \Phi}{\partial u}\big(u(t), v(t)\big) \cdot N\big(u(t), v(t)\big)\right)\bigg|_{t=0} - \frac{\partial \Phi}{\partial u}(u_0, v_0) \cdot \frac{\partial N}{\partial u}(u_0, v_0),$$

$$\frac{\partial^2 \Phi}{\partial u \partial v}(u_0, v_0) \cdot N(u_0, v_0) \\ = \frac{d}{dt}\left(\frac{\partial \Phi}{\partial u}\big(u(t), v(t)\big) \cdot N\big(u(t), v(t)\big)\right)\bigg|_{t=0} - \frac{\partial \Phi}{\partial u}(u_0, v_0) \cdot \frac{\partial N}{\partial v}(u_0, v_0),$$

$$\frac{\partial^2 \Phi}{\partial u \partial v}(u_0, v_0) \cdot N(u_0, v_0) = \frac{\partial^2 \Phi}{\partial v \partial u}(u_0, v_0) \cdot N(u_0, v_0) \\ = \frac{d}{dt}\left(\frac{\partial \Phi}{\partial v}\big(u(t), v(t)\big) \cdot N\big(u(t), v(t)\big)\right)\bigg|_{t=0} - \frac{\partial \Phi}{\partial v}(u_0, v_0) \cdot \frac{\partial N}{\partial u}(u_0, v_0),$$

$$\frac{\partial^2 \Phi}{\partial v^2}(u_0, v_0) \cdot N(u_0, v_0) \\ = \frac{d}{dt}\left(\frac{\partial \Phi}{\partial v}\big(u(t), v(t)\big) \cdot N\big(u(t), v(t)\big)\right)\bigg|_{t=0} - \frac{\partial \Phi}{\partial v}(u_0, v_0) \cdot \frac{\partial N}{\partial v}(u_0, v_0).$$

Since $N(u,v) = \frac{\partial \Phi}{\partial u}(u,v) \times \frac{\partial \Phi}{\partial v}(u,v) / \left\| \frac{\partial \Phi}{\partial u}(u,v) \times \frac{\partial \Phi}{\partial v}(u,v) \right\|$, we have that $N\big(u(t), v(t)\big) \perp \frac{\partial \Phi}{\partial u}\big(u(t), v(t)\big)$ and $N\big(u(t), v(t)\big) \perp \frac{\partial \Phi}{\partial v}\big(u(t), v(t)\big)$. So the above equations become:

$Equations$ 4.4.3: $$\frac{\partial^2 \Phi}{\partial u^2}(u_0, v_0) \cdot N(u_0, v_0) = -\frac{\partial \Phi}{\partial u}(u_0, v_0) \cdot \frac{\partial N}{\partial u}(u_0, v_0),$$

$$\frac{\partial^2 \Phi}{\partial u \partial v}(u_0, v_0) \cdot N(u_0, v_0) = -\frac{\partial \Phi}{\partial u}(u_0, v_0) \cdot \frac{\partial N}{\partial v}(u_0, v_0),$$

$$\frac{\partial^2 \Phi}{\partial u \partial v}(u_0, v_0) \cdot N(u_0, v_0) = -\frac{\partial \Phi}{\partial v}(u_0, v_0) \cdot \frac{\partial N}{\partial u}(u_0, v_0),$$

$$\frac{\partial^2 \Phi}{\partial v^2}(u_0, v_0) \cdot N(u_0, v_0) = -\frac{\partial \Phi}{\partial v}(u_0, v_0) \cdot \frac{\partial N}{\partial v}(u_0, v_0).$$

This means that we can rewrite the above equation for the component of curvature $k_{\gamma(t)}(N)\big|_{t=0}$ as (here I will again drop writing the arguments of the partials of Φ and N. They are being evaluated at (u_0, v_0) as usual):

$$k_{\gamma(t)}(N)\big|_{t=0} = \big(u'(0)\big)^2 \frac{\partial^2 \Phi}{\partial u^2} \cdot N + 2\big(u'(0)\big)\big(v'(0)\big) \frac{\partial^2 \Phi}{\partial u \partial v} \cdot N + \big(v'(0)\big)^2 \frac{\partial^2 \Phi}{\partial v^2} \cdot N$$

$$= \big(u'(0)\big)^2 \frac{\partial^2 \Phi}{\partial u^2} \cdot N + \big(u'(0)\big)\big(v'(0)\big) \frac{\partial^2 \Phi}{\partial u \partial v} \cdot N + \big(u'(0)\big)\big(v'(0)\big) \frac{\partial^2 \Phi}{\partial u \partial v} \cdot N$$

$$+\left(v'(0)\right)^2 \frac{\partial^2 \Phi}{\partial v^2} \cdot N$$

$$= -\left(u'(0)\right)^2 \frac{\partial \Phi}{\partial u} \cdot \frac{\partial N}{\partial u} - \left(u'(0)\right)\left(v'(0)\right) \frac{\partial \Phi}{\partial u} \cdot \frac{\partial N}{\partial v} - \left(u'(0)\right)\left(v'(0)\right) \frac{\partial \Phi}{\partial v} \cdot \frac{\partial N}{\partial u}$$

$$-\left(v'(0)\right)^2 \frac{\partial \Phi}{\partial v} \cdot \frac{\partial N}{\partial v}$$

$$= -\left(u'(0) \frac{\partial N}{\partial u} + v'(0) \frac{\partial N}{\partial v}\right) \cdot \left(u'(0) \frac{\partial \Phi}{\partial u} + v'(0) \frac{\partial \Phi}{\partial v}\right)$$

(where remember, · between to vectors means a vector dot product). So, we have that:

$$k_{\gamma(t)}(N)\big|_{t=0} = -\langle u'(0) \frac{\partial N}{\partial u} + v'(0) \frac{\partial N}{\partial v}, u'(0) \frac{\partial \Phi}{\partial u} + v'(0) \frac{\partial \Phi}{\partial v} \rangle.$$

Before, our equation for the component of curvature $k_{\gamma(t)}(N)\big|_{t=0}$ involved the second partials of the surface parametrization Φ but notice that in this equation were able to get rid of those second order partials. Those second partials in fact transferred to the partials of N and this will be crucial to us because this will allow us to rewrite the above quantity in a quadratic matrix form in a basis for the tangent plane $T_p(S)$ (which the first partials of Φ span). It's interesting to note that the vector in the second entry of the above inner product is the vector $\gamma'(0)$:

$$\gamma'(0) = \frac{d}{dt}\left(\gamma(t)\right)\Big|_{t=0} = \frac{d}{dt}\left(\Phi\left(u(t), v(t)\right)\right)\Big|_{t=0} = \frac{\partial \Phi}{\partial u} u'(0) + \frac{\partial \Phi}{\partial v} v'(0).$$

Now, we want to turn the above equation for $k_{\gamma(t)}\left(N(u_0, v_0)\right)\big|_{t=0}$ into a quadratic matrix form. This can be achieved by writing the differential of the Gauss map N in the basis $\frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial v}$ of the tangent plane $T_p(S)$. Notice that the fact that $\|N(u, v)\|$ is always one implies that:

$$0 = \frac{\partial}{\partial u}(\|N(u, v)\|^2) = \frac{\partial}{\partial u}\left(N(u, v) \cdot N(u, v)\right) = 2 \frac{\partial N}{\partial u}(u, v) \cdot N(u, v),$$

$$0 = \frac{\partial}{\partial v}(\|N(u, v)\|^2) = \frac{\partial}{\partial v}\left(N(u, v) \cdot N(u, v)\right) = 2 \frac{\partial N}{\partial v}(u, v) \cdot N(u, v).$$

This tells us that the partials $\frac{\partial N}{\partial u}(u, v)$ and $\frac{\partial N}{\partial v}(u, v)$ are always perpendicular to the normal vector $N(u, v)$ to the surface. $\frac{\partial N}{\partial u}(u_0, v_0)$ and $\frac{\partial N}{\partial v}(u_0, v_0)$ being perpendicular to the normal vector $N(u_0, v_0)$ implies that these two vectors lie in the tangent plane $T_p(S)$. Since $\frac{\partial \Phi}{\partial u}(u_0, v_0)$ and $\frac{\partial \Phi}{\partial v}(u_0, v_0)$ span $T_p(S)$ (Theorem 4.2.11), we get that there exists some $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ such that:

$$\frac{\partial N}{\partial u}(u_0, v_0) = a_{11} \frac{\partial \Phi}{\partial u}(u_0, v_0) + a_{21} \frac{\partial \Phi}{\partial v}(u_0, v_0),$$

$$\frac{\partial N}{\partial v}(u_0, v_0) = a_{12}\frac{\partial \Phi}{\partial u}(u_0, v_0) + a_{22}\frac{\partial \Phi}{\partial v}(u_0, v_0).$$

So the above equation for $k_{\gamma(t)}(N)\big|_{t=0}$ can be rewritten as:

$$\textit{Equation } 4.4.4\text{:} \quad k_{\gamma(t)}(N)\big|_{t=0} = -\langle\big(a_{11}u'(0) + a_{12}v'(0)\big)\frac{\partial \Phi}{\partial u} + \big(a_{21}u'(0) + a_{22}v'(0)\big)\frac{\partial \Phi}{\partial v},$$

$$u'(0)\frac{\partial \Phi}{\partial u} + v'(0)\frac{\partial \Phi}{\partial v}\rangle.$$

Notice that if we make the convention that the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ denotes a vector in the basis $\left\{\frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial v}\right\}$ (meaning the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ denotes the vector $a\frac{\partial \Phi}{\partial u} + b\frac{\partial \Phi}{\partial v}$), then the above equation can be rewritten as:

$$k_{\gamma(t)}(N)\big|_{t=0} = -\langle\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix}, \begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix}\rangle$$

in the basis $\left\{\frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial v}\right\}$. Or, since I like to write matrices on the right:

$$k_{\gamma(t)}(N)\big|_{t=0} = -\langle\begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix}\rangle.$$

Be careful not to confuse this inner product as a normal Euclidean inner product in the basis $(1,0)$ and $(0,1)$. This inner product is written in the basis $\left\{\frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial v}\right\}$ and so to evaluate this inner product you need to use Equation 4.4.4, which is written in the normal Euclidian inner product. So we were able to write an equation for $k_{\gamma(t)}(N)$ in a quadratic matrix form in the basis $\left\{\frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial v}\right\}$ (which by the way is a basis for the tangent plane $T_p(S)$). The interpretation of the above equation is that in the basis $\left\{\frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial v}\right\}$, the component of curvature of any unit speed curve $\gamma(t)$ that passes through p at time $t = 0$ with the tangential vector $\gamma'(0) = \frac{\partial \Phi}{\partial u}u'(0) + \frac{\partial \Phi}{\partial v}v'(0)$ in the unit normal direction N at p is given by the above quadratic matrix form. So as before, in order to get the minimum and maximum components of curvature in the unit directions $\gamma'(0)$, we have to look at the eigenvalues λ_1 and λ_2 of the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Since the Gaussian and mean curvatures are defined as the product and average of these eigenvalues respectively, we get that the Gaussian and mean curvatures of S at p are defined as:

$$K = \lambda_1\lambda_2 = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right),$$

$$H = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}trace\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right),$$

respectively. Ok, these equations might be great but this formula does not yet give us an explicit way to calculate the Gaussian and mean curvatures in terms of the surface parametrization Φ.

For that we need to find an explicit equation for the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in terms of the surface parametrization. So let's do that!

The coefficients $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ were defined as (here, as usual, I am omitting the arguments of the partials of Φ and N since we're focusing on the point p. They are being evaluated at (u_0, v_0)):

$$\frac{\partial N}{\partial u} = a_{11}\frac{\partial \Phi}{\partial u} + a_{21}\frac{\partial \Phi}{\partial v},$$
$$\frac{\partial N}{\partial v} = a_{12}\frac{\partial \Phi}{\partial u} + a_{22}\frac{\partial \Phi}{\partial v}.$$

If we dot both sides of both equations by $\frac{\partial \Phi}{\partial u}$ we get that:

$$\frac{\partial N}{\partial u}\cdot\frac{\partial \Phi}{\partial u} = a_{11}\frac{\partial \Phi}{\partial u}\cdot\frac{\partial \Phi}{\partial u} + a_{21}\frac{\partial \Phi}{\partial v}\cdot\frac{\partial \Phi}{\partial u},$$
$$\frac{\partial N}{\partial v}\cdot\frac{\partial \Phi}{\partial u} = a_{12}\frac{\partial \Phi}{\partial u}\cdot\frac{\partial \Phi}{\partial u} + a_{22}\frac{\partial \Phi}{\partial v}\cdot\frac{\partial \Phi}{\partial u}.$$

And if we dot both sides of the previous equations by $\frac{\partial \Phi}{\partial v}$ we get that:

$$\frac{\partial N}{\partial u}\cdot\frac{\partial \Phi}{\partial v} = a_{11}\frac{\partial \Phi}{\partial u}\cdot\frac{\partial \Phi}{\partial v} + a_{21}\frac{\partial \Phi}{\partial v}\cdot\frac{\partial \Phi}{\partial v},$$
$$\frac{\partial N}{\partial v}\cdot\frac{\partial \Phi}{\partial v} = a_{12}\frac{\partial \Phi}{\partial u}\cdot\frac{\partial \Phi}{\partial v} + a_{22}\frac{\partial \Phi}{\partial v}\cdot\frac{\partial \Phi}{\partial v}.$$

Together these two systems of equations give:

$$\frac{\partial N}{\partial u}\cdot\frac{\partial \Phi}{\partial u} = a_{11}\frac{\partial \Phi}{\partial u}\cdot\frac{\partial \Phi}{\partial u} + a_{21}\frac{\partial \Phi}{\partial v}\cdot\frac{\partial \Phi}{\partial u},$$
$$\frac{\partial N}{\partial v}\cdot\frac{\partial \Phi}{\partial u} = a_{12}\frac{\partial \Phi}{\partial u}\cdot\frac{\partial \Phi}{\partial u} + a_{22}\frac{\partial \Phi}{\partial v}\cdot\frac{\partial \Phi}{\partial u},$$
$$\frac{\partial N}{\partial u}\cdot\frac{\partial \Phi}{\partial v} = a_{11}\frac{\partial \Phi}{\partial u}\cdot\frac{\partial \Phi}{\partial v} + a_{21}\frac{\partial \Phi}{\partial v}\cdot\frac{\partial \Phi}{\partial v},$$
$$\frac{\partial N}{\partial v}\cdot\frac{\partial \Phi}{\partial v} = a_{12}\frac{\partial \Phi}{\partial u}\cdot\frac{\partial \Phi}{\partial v} + a_{22}\frac{\partial \Phi}{\partial v}\cdot\frac{\partial \Phi}{\partial v}.$$

Mathematicians are evidently too lazy to write out everything explicitly in differential geometry and so they assigned the following letters to denote the following quantities:

$$E(u,v) = \frac{\partial \Phi}{\partial u}(u,v)\cdot\frac{\partial \Phi}{\partial u}(u,v), \qquad F(u,v) = \frac{\partial \Phi}{\partial u}(u,v)\cdot\frac{\partial \Phi}{\partial v}(u,v) = \frac{\partial \Phi}{\partial v}(u,v)\cdot\frac{\partial \Phi}{\partial u}(u,v),$$
$$G(u,v) = \frac{\partial \Phi}{\partial v}(u,v)\cdot\frac{\partial \Phi}{\partial v}(u,v),$$

$$e(u,v) = \frac{\partial N}{\partial u}(u,v) \cdot \frac{\partial \Phi}{\partial u}(u,v), \qquad f(u,v) = \frac{\partial N}{\partial u}(u,v) \cdot \frac{\partial \Phi}{\partial v}(u,v) = \frac{\partial N}{\partial v}(u,v) \cdot \frac{\partial \Phi}{\partial u}(u,v),$$

$$g(u,v) = \frac{\partial N}{\partial v}(u,v) \cdot \frac{\partial \Phi}{\partial v}(u,v)$$

(the fact that $\frac{\partial \Phi}{\partial u}(u,v) \cdot \frac{\partial \Phi}{\partial v}(u,v) = \frac{\partial \Phi}{\partial v}(u,v) \cdot \frac{\partial \Phi}{\partial u}(u,v)$ comes from the commutativity of the dot product and the fact that $\frac{\partial N}{\partial u}(u,v) \cdot \frac{\partial \Phi}{\partial v}(u,v) = \frac{\partial N}{\partial v}(u,v) \cdot \frac{\partial \Phi}{\partial u}(u,v)$ can be seen from the two equations in the middle in Equations 4.4.3). The functions E, F, G are called the "first fundamental form" since they only involve the first partials of Φ. And the functions e, f, g are called the "second fundamental form" since they involve the second order partials of Φ through Equations 4.4.3. The above definitions now allow use to rewrite the previous system of equations in the nice form (here I omit the arguments of E, F, G, e, f, g since we're concentrating on the point p. They're being evaluated at (u_0, v_0)):

$$e = a_{11}E + a_{21}F,$$

$$f = a_{12}E + a_{22}F,$$

$$f = a_{11}F + a_{21}G,$$

$$g = a_{12}F + a_{22}G.$$

Notice that this system of equations can be rewritten in the matrix form:

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Multiply both sides from the left by the inverse of $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ to get that:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix}$$

One might be worried that the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ is not invertible, but it is since using the Pythagorean theorem for vectors,

$$\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\|^2 + \left(\frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v} \right)^2 = \left\| \frac{\partial \Phi}{\partial u} \right\|^2 \left\| \frac{\partial \Phi}{\partial v} \right\|^2,$$

or in other words:

$$\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\|^2 + F^2 = EG,$$

we see that:

$$\det\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix} \right) = EG - F^2 = \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\|^2 \neq 0$$

by condition 3 of a surface parametrization. So the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ is always invertible and using the formula for the inverse of a two by two matrix, we have that:

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \frac{1}{EG - F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$$

So the above equation for $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ can be rewritten as:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{EG - F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix}\begin{bmatrix} e & f \\ f & g \end{bmatrix}$$

Thus we finally have an explicit equation for the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in terms of the surface parametrization Φ (notice that all of the coefficients E, F, G, e, f, g are defined directly through the surface parametrization). Notice that this gives us the system of equations:

$$a_{11} = \frac{eG - fF}{EG - F^2}, \qquad a_{12} = \frac{fG - gF}{EG - F^2}$$
$$a_{21} = \frac{fE - eF}{EG - F^2}, \qquad a_{22} = \frac{gE - fF}{EG - F^2}$$

This allows us to explicitly calculate the Gaussian and mean curvatures of S at p in terms of the first and second fundamental form:

$$K = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{21}a_{12} = \frac{eg - f^2}{EG - F^2},$$
$$H = \frac{1}{2} trace\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2} \cdot \frac{eG - 2fF + gE}{EG - F^2}$$

With this we finally arrive at the formal definition of the curvatures of surfaces.

Definition 4.4.5: *Let S be a smooth 2-dimensional surface sitting in $\mathbb{R}^3$ and $p \in S$ be any point on it. By the definition of a smooth surface, we know that there exists some surface parametrization Φ of S at p. Let $(u_0, v_0) = \Phi^{-1}(p)$. Let the Gauss map be the function:*

$$N(u, v) = \frac{\frac{\partial \Phi}{\partial u}(u, v) \times \frac{\partial \Phi}{\partial v}(u, v)}{\left\| \frac{\partial \Phi}{\partial u}(u, v) \times \frac{\partial \Phi}{\partial v}(u, v) \right\|}.$$

*The **first fundamental form** of S is defined as:*

$$E(u, v) = \frac{\partial \Phi}{\partial u}(u, v) \cdot \frac{\partial \Phi}{\partial u}(u, v), \qquad F(u, v) = \frac{\partial \Phi}{\partial u}(u, v) \cdot \frac{\partial \Phi}{\partial v}(u, v) = \frac{\partial \Phi}{\partial v}(u, v) \cdot \frac{\partial \Phi}{\partial u}(u, v),$$
$$G(u, v) = \frac{\partial \Phi}{\partial v}(u, v) \cdot \frac{\partial \Phi}{\partial v}(u, v).$$

*And the **second fundamental form** of S is defined as:*

$$e(u,v) = \frac{\partial N}{\partial u}(u,v) \cdot \frac{\partial \Phi}{\partial u}(u,v), \qquad f(u,v) = \frac{\partial N}{\partial u}(u,v) \cdot \frac{\partial \Phi}{\partial v}(u,v) = \frac{\partial N}{\partial v}(u,v) \cdot \frac{\partial \Phi}{\partial u}(u,v),$$

$$g(u,v) = \frac{\partial N}{\partial v}(u,v) \cdot \frac{\partial \Phi}{\partial v}(u,v).$$

Then the ***Gaussian and mean curvatures*** *of S at the point p are defined as:*

$$K = \frac{e(u_0,v_0)g(u_0,v_0) - \left(f(u_0,v_0)\right)^2}{E(u_0,v_0)G(u_0,v_0) - \left(F(u_0,v_0)\right)^2},$$

$$H = \frac{1}{2} \cdot \frac{e(u_0,v_0)G(u_0,v_0) - 2f(u_0,v_0)F(u_0,v_0) + g(u_0,v_0)E(u_0,v_0)}{E(u_0,v_0)G(u_0,v_0) - \left(F(u_0,v_0)\right)^2}.$$

respectively. If we omit the arguments of e, f, g, E, F, G in the above equation, then the above equations take the nice to look at form:

$$K = \frac{eg - f^2}{EG - F^2},$$

$$H = \frac{1}{2} \cdot \frac{eG - 2fF + gE}{EG - F^2}.$$

As a point of interest, Equations 4.4.3 gives us that the second fundamental form is also given by:

$$e(u,v) = -N(u,v) \cdot \frac{\partial^2 \Phi}{\partial u^2}(u,v),$$

$$f(u,v) = -N(u,v) \cdot \frac{\partial^2 \Phi}{\partial u \partial v}(u,v),$$

$$g(u,v) = -N(u,v) \cdot \frac{\partial^2 \Phi}{\partial v^2}(u,v),$$

which are often easier to calculate than the above alternative equations for e, f, g.

One issue that does arise in the above definitions of the Gaussian and mean curvatures is the issue of whether they are well defined. Indeed, there might be two surface parametrizations Φ and Ψ of S at p and how do we know that the above formulas for the Gaussian and mean curvatures give the same value no matter which surface parametrization you use to calculate them. The answer is that that the values of the Gaussian curvature will always turn out to be the same no matter which surface parametrization you use to calculate it and the same goes for the mean curvature up to sign. The reason for this lies in the fact that the Gaussian and mean curvatures are quantities obtained from geometric principles which are independent of surface parametrizations. Indeed, if you look back and see how we arrived at the definition of the Gaussian and mean curvatures you'll see that we looked at which unit directions $\gamma'(0)$ minimize and maximize the component of curvature quantity $k_{\gamma(t)}(N)\big|_{t=0}$ at our point and such surface curves $\gamma(t)$ are completely geometric objects that have no reference to surface parametrizations. The fact that the mean curvature can change sign arises from the fact that the unit normal vector N that you consider at your point might flip direction as you change from one surface parametrization to the other and thus might flip the sign of these component of curvatures. In explanation, in the Φ surface parametrization the unit normal vector that we consider is $\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}/\left\|\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}\right\|$ and in the Ψ surface parametrization we consider the unit normal vector $\frac{\partial\Psi}{\partial u}\times\frac{\partial\Psi}{\partial v}/\left\|\frac{\partial\Psi}{\partial u}\times\frac{\partial\Psi}{\partial v}\right\|$, which might point in the opposite direction (at every point of a surface there are always to unit normal vectors). The Gaussian curvature doesn't ever slip sign under a change of surface parametrizations since it is the product of two (an even number) of such component of curvatures that can potential change sign only together.

Let's calculate some specific examples of surface curvatures. In the next example we derive the extremely important formula for the Gaussian curvature and mean curvature of a surface parametrized in a graph surface parametrization.

Example 4.4.5: Suppose that S is the graph of a function $z = f(x, y)$. Let's calculate the Gaussian and mean curvatures of S at some point $p = \big(x_0, y_0, f(x_0, y_0)\big)$ on it. The graph surface parametrization that parametrizes this surface is given by:

$$\Phi(x, y) = \big(x, y, f(x, y)\big).$$

Notice that $p = \big(x_0, y_0, f(x_0, y_0)\big) = \Phi(x_0, y_0)$. Let calculate the Gauss map and the first and second fundamental forms of Φ at (x_0, y_0). We have that:

$$N(x_0, y_0) = \frac{\frac{\partial\Phi}{\partial x}(x_0, y_0)\times\frac{\partial\Phi}{\partial y}(x_0, y_0)}{\left\|\frac{\partial\Phi}{\partial x}(x_0, y_0)\times\frac{\partial\Phi}{\partial y}(x_0, y_0)\right\|} = \frac{\begin{bmatrix}1\\0\\\frac{\partial f}{\partial x}(x_0, y_0)\end{bmatrix}\times\begin{bmatrix}0\\1\\\frac{\partial f}{\partial y}(x_0, y_0)\end{bmatrix}}{\left\|\begin{bmatrix}1\\0\\\frac{\partial f}{\partial x}(x_0, y_0)\end{bmatrix}\times\begin{bmatrix}0\\1\\\frac{\partial f}{\partial y}(x_0, y_0)\end{bmatrix}\right\|}$$

$$= \frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial x}(x_0, y_0)\right)^2+\left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^2}}\begin{bmatrix}-\frac{\partial f}{\partial x}(x_0, y_0)\\-\frac{\partial f}{\partial y}(x_0, y_0)\\1\end{bmatrix}$$

Notice that this is just a normalized version of the equation for a perpendicular vector to the surface that we got at the end of Section 2. Now, the partials of Φ are given by:

$$\frac{\partial\Phi}{\partial x}(x_0, y_0) = \begin{bmatrix}1\\0\\\frac{\partial f}{\partial x}(x_0, y_0)\end{bmatrix}, \qquad \frac{\partial\Phi}{\partial y}(x_0, y_0) = \begin{bmatrix}0\\1\\\frac{\partial f}{\partial y}(x_0, y_0)\end{bmatrix}$$

$$\frac{\partial^2\Phi}{\partial x^2}(x_0,y_0)=\begin{bmatrix}0\\0\\\frac{\partial^2 f}{\partial x^2}(x_0,y_0)\end{bmatrix},\qquad \frac{\partial^2\Phi}{\partial x\partial y}(x_0,y_0)=\begin{bmatrix}0\\0\\\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)\end{bmatrix},$$

$$\frac{\partial^2\Phi}{\partial y^2}(x_0,y_0)=\begin{bmatrix}0\\0\\\frac{\partial^2 f}{\partial y^2}(x_0,y_0)\end{bmatrix}$$

So the first and second fundamental forms are given by:

$$E(x_0,y_0)=\frac{\partial\Phi}{\partial x}(x_0,y_0)\cdot\frac{\partial\Phi}{\partial x}(x_0,y_0)=1+\left(\frac{\partial f}{\partial x}(x_0,y_0)\right)^2,$$

$$F(x_0,y_0)=\frac{\partial\Phi}{\partial x}(x_0,y_0)\cdot\frac{\partial\Phi}{\partial y}(x_0,y_0)=\frac{\partial f}{\partial x}(x_0,y_0)\frac{\partial f}{\partial y}(x_0,y_0),$$

$$G(x_0,y_0)=\frac{\partial\Phi}{\partial y}(x_0,y_0)\cdot\frac{\partial\Phi}{\partial y}(x_0,y_0)=1+\left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^2,$$

$$e(x_0,y_0)=-N(x_0,y_0)\cdot\frac{\partial^2\Phi}{\partial x^2}(x_0,y_0)=-\frac{\frac{\partial^2 f}{\partial x^2}(x_0,y_0)}{\sqrt{1+\left(\frac{\partial f}{\partial x}(x_0,y_0)\right)^2+\left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^2}},$$

$$f(x_0,y_0)=-N(x_0,y_0)\cdot\frac{\partial^2\Phi}{\partial x\partial y}(x_0,y_0)=-\frac{\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)}{\sqrt{1+\left(\frac{\partial f}{\partial x}(x_0,y_0)\right)^2+\left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^2}},$$

$$g(x_0,y_0)=-N(x_0,y_0)\cdot\frac{\partial^2\Phi}{\partial y^2}(x_0,y_0)=-\frac{\frac{\partial^2 f}{\partial y^2}(x_0,y_0)}{\sqrt{1+\left(\frac{\partial f}{\partial x}(x_0,y_0)\right)^2+\left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^2}}.$$

So, by Definition 4.4.5 we get that the Gaussian and mean curvatures of S at $p=\left(x_0,y_0,f(x_0,y_0)\right)$ are (here I omit the arguments of the partials of f to make the equations shorter. They are being evaluated at (x_0,y_0)):

$$K = \frac{\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2}{\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)^2},$$

$$H = -\frac{1}{2} \cdot \frac{\frac{\partial^2 f}{\partial x^2}\left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right) - 2\frac{\partial^2 f}{\partial x \partial y} \cdot \frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y^2}\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)}{\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)^{\frac{3}{2}}},$$

respectively.

Let's compute some classical surface curvatures.

Example 4.4.6: Let's calculate the Gaussian and mean curvatures of the plane. Suppose that we have a plane that is the graph of $z = ax + by$ where a and b are some real constants (the calculation of the surface curvatures of a plane in the cases when you must use graph parametrizations of the form $y = f(x, z)$ or $x = f(y, z)$ are similar and give the same answer). Then, if we plug in $f(x, y) = ax + by$ into the formula for the surface curvatures derived in Example 4.4.5 we get that both the Gaussian and mean curvatures of the plane at any point are equal to zero:

$$K \equiv 0 \qquad and \qquad H \equiv 0$$

since all second partials of $f(x, y) = ax + by$ are zero. This makes sense since the plane is not "curving;" it's flat!

Example 4.4.7: Now let's calculate the Gaussian and mean curvatures of the sphere. Take any sphere of radius $r > 0$. Let's calculate its surface curvatures in the upper half hemisphere (the region of the sphere strictly above the x-y plane). The calculations of the surface curvatures of the sphere in the other hemisphere are similar and they turn out to give the same answer. The upper half hemisphere is the graph of the function $z = \sqrt{r^2 - x^2 - y^2}$. Plugging in $f(x, y)$ into the equations for the surface curvatures derived in Example 4.4.5 gives that at any point in the upper half hemisphere,

$$K(x, y) = \frac{\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2}{\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)^2} = \frac{\frac{y^2 - r^2}{\sqrt{r^2 - x^2 - y^2}^3} \cdot \frac{x^2 - r^2}{\sqrt{r^2 - x^2 - y^2}^3} - \left(\frac{-xy}{\sqrt{r^2 - x^2 - y^2}^3}\right)^2}{\left(1 + \left(\frac{-x}{\sqrt{r^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{r^2 - x^2 - y^2}}\right)^2\right)^2}$$

$$= \frac{\frac{r^4 - r^2x^2 - r^2y^2}{(r^2 - x^2 - y^2)^3}}{\frac{r^4}{(r^2 - x^2 - y^2)^2}} = \frac{r^2(r^2 - x^2 - y^2)}{r^4(r^2 - x^2 - y^2)} = \frac{1}{r^2},$$

and (I broke the fraction up into three pieces in the second equality below so that it is easier to fit the equations on the page):

$$H(x,y) = -\frac{1}{2}\frac{\frac{\partial^2 f}{\partial x^2}\left(1+\left(\frac{\partial f}{\partial y}\right)^2\right) - 2\frac{\partial^2 f}{\partial x \partial y}\cdot\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y^2}\left(1+\left(\frac{\partial f}{\partial x}\right)^2\right)}{\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^{\frac{3}{2}}}$$

$$= -\frac{1}{2}\frac{\frac{\partial^2 f}{\partial x^2}\left(1+\left(\frac{\partial f}{\partial y}\right)^2\right)}{\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^{\frac{3}{2}}} + \frac{\frac{\partial^2 f}{\partial x \partial y}\cdot\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}}{\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^{\frac{3}{2}}} - \frac{1}{2}\frac{\frac{\partial^2 f}{\partial y^2}\left(1+\left(\frac{\partial f}{\partial x}\right)^2\right)}{\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)^{\frac{3}{2}}}$$

$$= -\frac{1}{2}\frac{\frac{y^2-r^2}{\sqrt{r^2-x^2-y^2}^3}\left(1+\left(\frac{-y}{\sqrt{r^2-x^2-y^2}}\right)^2\right)}{\left(1+\left(\frac{-x}{\sqrt{r^2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{r^2-x^2-y^2}}\right)^2\right)^{\frac{3}{2}}}$$

$$+\frac{\frac{-xy}{\sqrt{r^2-x^2-y^2}^3}\frac{-x}{\sqrt{r^2-x^2-y^2}}\frac{-y}{\sqrt{r^2-x^2-y^2}}}{\left(1+\left(\frac{-x}{\sqrt{r^2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{r^2-x^2-y^2}}\right)^2\right)^{\frac{3}{2}}}$$

$$-\frac{1}{2}\frac{\frac{x^2-r^2}{\sqrt{r^2-x^2-y^2}^3}\left(1+\left(\frac{-x}{\sqrt{r^2-x^2-y^2}}\right)^2\right)}{\left(1+\left(\frac{-x}{\sqrt{r^2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{r^2-x^2-y^2}}\right)^2\right)^{\frac{3}{2}}}$$

$$= -\frac{1}{2}\frac{\frac{r^2x^2+r^2y^2-x^2y^2-r^4}{\sqrt{r^2-x^2-y^2}^5}}{\frac{r^3}{\sqrt{r^2-x^2-y^2}^3}} + \frac{\frac{x^2y^2}{\sqrt{r^2-x^2-y^2}^5}}{\frac{r^3}{\sqrt{r^2-x^2-y^2}^3}} - \frac{1}{2}\frac{\frac{r^2x^2+r^2y^2-x^2y^2-r^4}{\sqrt{r^2-x^2-y^2}^5}}{\frac{r^3}{\sqrt{r^2-x^2-y^2}^3}}$$

$$= -\frac{r^2x^2+r^2y^2-r^4}{r^3(r^2-x^2-y^2)} = -\frac{r^2(x^2+y^2-r^2)}{r^3(r^2-x^2-y^2)} = \frac{1}{r}.$$

And so we get that the Gaussian and mean curvatures in the upper half hemisphere are:

$$K(x,y) = \frac{1}{r^2},$$

$$H(x,y) = \frac{1}{r}.$$

If you do a similar sort of calculation but in the other hemispheres you will get the same exact answers. So, we have that the Gaussian and mean curvatures on the sphere are constantly:

$$K \equiv \frac{1}{r^2} \qquad and \qquad H \equiv \frac{1}{r}.$$

It should be clear that the Gaussian and mean curvatures of the sphere are the same at any point since the sphere is completely symmetric and the surface curvatures are invariant under rotations (since they were arrived at using geometric quantities such as component of curvatures). If we cite this realization in the beginning of the calculation, then we could have used the above $f(x,y)$ to calculate the surface curvatures just at the north pole $(0,0,r)$ and then said that the answer that we got applied to every point on the sphere. The reason why this would have been useful is that the calculation of the surface curvatures at the north pole is much shorter since all of the quantities $x, y, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are zero and so one could avoid the above lengthy algebraic manipulations. Indeed if you calculate directly the surface curvatures of the sphere at the north pole you get that:

$$K(0,0) = \frac{\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2}{\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)^2} = \frac{\frac{0^2 - r^2}{\sqrt{r^2 - 0^2 - 0^2}^3} \cdot \frac{0^2 - r^2}{\sqrt{r^2 - 0^2 - 0^2}^3} - \left(\frac{-0 \cdot 0}{\sqrt{r^2 - 0^2 - o^2}^3}\right)^2}{(1 + 0^2 + 0^2)^2}$$

$$= \frac{1}{r^2},$$

and:

$$H(0,0) = -\frac{1}{2} \frac{\frac{\partial^2 f}{\partial x^2}\left(1 + \left(\frac{\partial f}{\partial y}\right)^2\right) - 2\frac{\partial^2 f}{\partial x \partial y} \cdot \frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y^2}\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)}{\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)^{\frac{3}{2}}}$$

$$= -\frac{1}{2} \frac{\frac{0^2 - r^2}{\sqrt{r^2 - 0^2 - 0^2}^3}\left(1 + \left(\frac{-0}{\sqrt{r^2 - 0^2 - 0^2}}\right)^2\right)}{(1 + 0^2 + 0^2)^{\frac{3}{2}}}$$

$$+\frac{\frac{-0\cdot 0}{\sqrt{r^2-0^2-0^2}^3}\frac{-0}{\sqrt{r^2-0^2-0^2}}\frac{-0}{\sqrt{r^2-0^2-0^2}}}{(1+0^2+0^2)^{\frac{3}{2}}}-\frac{1}{2}\frac{\frac{0^2-r^2}{\sqrt{r^2-0^2-0^2}^3}\left(1+\left(\frac{-0}{\sqrt{r^2-0^2-0^2}}\right)^2\right)}{(1+0^2+0^2)^{\frac{3}{2}}}$$

$$=\frac{1}{r}.$$

And so by the symmetry argument mentioned above, we get that the Gaussian and mean curvatures of the sphere are constantly $1/r^2$ and $1/r$ respectively. Although the difference in the length of the above calculation when this trick was and was not used is not too breathtaking, this trick of rotating something into a convenient position so that the calculation becomes much more elegant is an important trick in variational differential geometry and we will encounter a more dramatic example of this in Chapter 8 when we do calculations involving minimal surfaces and total Gaussian curvatures in n-dimensional space. Tricks that help shorten calculations in differential geometry are often sought after as this field is often characterized by long and difficult computations.

The determination of the surface curvatures of the plane and the sphere can in fact be done directly by geometric arguments. Since none of the unit speed surface curves of the plane have any component of curvature in the unit normal direction to the plane, all principal curvatures at any point of the plane are equal to zero and thus so are the Gaussian and mean curvatures. A similar sort of thing goes for the sphere. By the symmetry of the sphere all of the principal curvatures are equal, and it is not hard to see that the component of curvature of any surface curve in the unit normal direction to the sphere at any point is equal to $-1/r$. The facts that the Gaussian and mean curvatures of the sphere are $1/r^2$ and $1/r$ respectively follow immediately from this using the geometric interpretation of the Gaussian and mean curvatures.

Section 5: The Metric Tensor

Two very important concepts that we will be studying the variational natures of in the next chapter are arc-length of surface curves and surface area of surfaces. And it turns out that both of these concepts are conveniently described by what's called the metric tensor which is a matrix that describes the metric properties of a surface locally at each point.

Let's suppose that we have a surface S and a surface parametrization Φ of this surface. Let $\big(u(t),v(t)\big)$ be some curve in Φ's domain. The curve $\gamma(t)=\Phi\big(u(t),v(t)\big)$ is a curve that lies on the surface S. We can now ask the question: what is the arc-length of the surface curve $\gamma(t)$ from time t_0 to time t (let's suppose that the interval $[t_0,t]$ is a subset of the domain of $\gamma(t)$). The answer is simple, from calculus we know that the arc-length of $\gamma(t)$ from t_0 to t is given by:

$$L(t)=\int_{t_0}^{t}\|\gamma'(t)\|dt=\int_{t_0}^{t}\sqrt{\gamma'(t)\cdot\gamma'(t)}dt=\int_{t_0}^{t}\sqrt{\frac{d}{dt}\Big(\Phi\big(u(t),v(t)\big)\Big)\cdot\frac{d}{dt}\Big(\Phi\big(u(t),v(t)\big)\Big)}\,dt$$

$$=$$

$$\int_{t_0}^{t} \sqrt{\left(\frac{\partial\Phi}{\partial u}\big(u(t),v(t)\big)u'(t) + \frac{\partial\Phi}{\partial v}\big(u(t),v(t)\big)v'(t)\right)\cdot\left(\frac{\partial\Phi}{\partial u}\big(u(t),v(t)\big)u'(t) + \frac{\partial\Phi}{\partial v}\big(u(t),v(t)\big)v'(t)\right)}\,dt$$

Now if you use the distributive property of the vector dot product $\cdot$, you will get that (here I am omitting the arguments of the partials of Φ to make the equations shorter. They are being evaluated at $\big(u(t),v(t)\big)$):

$$L(t) = \int_{t_0}^{t} \sqrt{\frac{\partial\Phi}{\partial u}\cdot\frac{\partial\Phi}{\partial u}\big(u'(t)\big)^2 + 2\frac{\partial\Phi}{\partial u}\cdot\frac{\partial\Phi}{\partial v}u'(t)v'(t) + \frac{\partial\Phi}{\partial v}\cdot\frac{\partial\Phi}{\partial v}\big(v'(t)\big)^2}\,dt$$

If we use the notation of the first fundamental form, we get that we can rewrite the above equation as:

$$L(t) = \int_{t_0}^{t} \sqrt{E\big(u(t),v(t)\big)\big(u'(t)\big)^2 + 2F\big(u(t),v(t)\big)u'(t)v'(t) + G\big(u(t),v(t)\big)\big(v'(t)\big)^2}\,dt.$$

Notice that this is a quadratic form and so this can be rewritten in a quadratic matrix form:

$$L(t) = \int_{t_0}^{t_1} \sqrt{\left\langle \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix}, \begin{bmatrix} E\big(u'(t),v'(t)\big) & F\big(u(t),v(t)\big) \\ F\big(u(t),v(t)\big) & G\big(u(t),v(t)\big) \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} \right\rangle}\,dt.$$

Or if we omit the arguments of u', v', E, F, G, we get that the above equation takes the nice to look at form:

$$L(t) = \int_{t_0}^{t_1} \sqrt{\left\langle \begin{bmatrix} u' \\ v' \end{bmatrix}, \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} \right\rangle}\,dt.$$

Thus from here we can see that the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ helps describe the length of surface curves. In fact, if we let dL denote the differential length of $\gamma(t)$ and let du and dv be the differential forms of $u'(t)dt$ and $v'(t)dt$, then the above equation implies that:

$$dL^2 = \left\langle \begin{bmatrix} du \\ dv \end{bmatrix}, \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix} \right\rangle.$$

Since the differential vector $\begin{bmatrix} du \\ dv \end{bmatrix}$ can point in any direction in $\mathbb{R}^2$, we get that $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ describes the differential length of vectors on the surface. In other words, at any point on the surface the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ describes the local metric of the surface. For this reason, the matrix:

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{bmatrix}.$$

is called the "metric tensor" of the surface. It turns out that the metric tensor also describes the differential surface area of a surface. Over any region Ω that is a subset of Φ's domain, from multivariable calculus we know that the surface area of the $\Phi[\Omega]$ portion of S is given by:

$$A = \iint_{\Omega} \left\| \frac{\partial \Phi}{\partial u}(u,v) \times \frac{\partial \Phi}{\partial v}(u,v) \right\| dudv.$$

By the Pythagorean theorem for vectors,

$$\left\| \frac{\partial \Phi}{\partial u}(u,v) \times \frac{\partial \Phi}{\partial v}(u,v) \right\|^2 + \left(\frac{\partial \Phi}{\partial u}(u,v) \cdot \frac{\partial \Phi}{\partial v}(u,v) \right)^2 = \left\| \frac{\partial \Phi}{\partial u}(u,v) \right\|^2 \left\| \frac{\partial \Phi}{\partial v}(u,v) \right\|^2,$$

or in other words:

$$\left\| \frac{\partial \Phi}{\partial u}(u,v) \times \frac{\partial \Phi}{\partial v}(u,v) \right\|^2 + \left(F(u,v) \right)^2 = E(u,v)G(u,v),$$

we get that we can rewrite the equation for the surface area of $\Phi[\Omega]$ above as:

$$A = \iint_{\Omega} \sqrt{E(u,v)G(u,v) - \left(F(u,v)\right)^2}\, dudv = \iint_{\Omega} \sqrt{\det\left(\begin{bmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{bmatrix} \right)}\, dudv$$

Or if we omit the arguments of E, F, G, we get that the above equation takes the nice to look at form:

$$A = \iint_{\Omega} \sqrt{\det\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix} \right)}\, dudv$$

So we get that the metric tensor describes the surface area of a surface locally by the differential formula:

$$dA = \sqrt{\det\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix} \right)}\, dudv$$

As we can see, the metric tensor plays a central role in the description of the local metric of a surface and its surface area. Note that even though the metric tensor describes geometric properties of the surface, the form of the metric tensor highly depends on which surface parametrization you use. If you change the surface parametrization, you will change the metric tensor.

Section 6: The Christoffel Symbols

In the study of the curvatures of surfaces, it is no surprise at all that we encounter equations that involve the second partials of surface parametrizations. And it turns out that throughout all of differential geometry one will encounter many equations involving the second partials of surface parametrizations and so order not to go insane while using such equations, mathematicians have invented symbols to represent many of those second partial derivaitve quantities. One such set of symbols are the Christoffel symbols and since we will be encountering such equation in the next chapters, we might as well define them here.

Let S be a smooth surface and let $\Phi(u, v)$ be a surface parametrization of S. Let $N(u, v)$ be the Gauss map defined by:

$$N(u,v) = \frac{\frac{\partial \Phi}{\partial u}(u,v) \times \frac{\partial \Phi}{\partial v}(u,v)}{\left\| \frac{\partial \Phi}{\partial u}(u,v) \times \frac{\partial \Phi}{\partial v}(u,v) \right\|}.$$

Let us consider the second order partials $\frac{\partial^2 \Phi}{\partial u^2}(u,v), \frac{\partial^2 \Phi}{\partial u \partial v}(u,v), \frac{\partial^2 \Phi}{\partial v^2}(u,v)$. These are vector in $\mathbb{R}^3$. Now, the list of vectors $\left\{\frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial v}, N = \frac{\partial \Phi}{\partial u}(u,v) \times \frac{\partial \Phi}{\partial v}(u,v) / \left\| \frac{\partial \Phi}{\partial u}(u,v) \times \frac{\partial \Phi}{\partial v}(u,v) \right\| \right\}$ is a linearly independent list by condition 3 of a surface parametrization and the properties of the vector cross product. So they form a basis of $\mathbb{R}^3$ and thus there exist (the number in the superscript of the Γ's below are indices and not powers):

$$\Gamma_{11}^1(u,v), \Gamma_{11}^2(u,v), \Gamma_{12}^1(u,v), \Gamma_{12}^2(u,v), \Gamma_{22}^1(u,v), \Gamma_{22}^2(u,v), a(u,v), b(u,v), c(u,v) \in \mathbb{R}$$

such that:

$$\frac{\partial^2 \Phi}{\partial u^2}(u,v) = \Gamma_{11}^1(u,v)\frac{\partial \Phi}{\partial u}(u,v) + \Gamma_{11}^2(u,v)\frac{\partial \Phi}{\partial v}(u,v) + a(u,v)N(u,v),$$

$$\frac{\partial^2 \Phi}{\partial u \partial v}(u,v) = \Gamma_{12}^1(u,v)\frac{\partial \Phi}{\partial u}(u,v) + \Gamma_{12}^2(u,v)\frac{\partial \Phi}{\partial v}(u,v) + b(u,v)N(u,v),$$

$$\frac{\partial^2 \Phi}{\partial v^2}(u,v) = \Gamma_{22}^1(u,v)\frac{\partial \Phi}{\partial u}(u,v) + \Gamma_{22}^2(u,v)\frac{\partial \Phi}{\partial v}(u,v) + c(u,v)N(u,v).$$

The coefficient functions $\Gamma_{11}^1(u,v), \Gamma_{11}^2(u,v), \Gamma_{12}^1(u,v), \Gamma_{12}^2(u,v), \Gamma_{22}^1(u,v), \Gamma_{22}^2(u,v)$ are called the **Christoffel Symbols** and they fundamentally describe how the surface (or more accurately the surface parametrization) curves into the tangent plane. The coefficients $a(u,v), b(u,v), c(u,v)$ are actually something we've seen before. Let's calculate what they are. Dot both sides of each equation above by $N(u,v)$. Since $N(u,v) \perp \frac{\partial \Phi}{\partial u}(u,v)$ and $N(u,v) \perp \frac{\partial \Phi}{\partial u}(u,v)$, all of the coefficients with the Christoffel symbols will go away and we will get that:

$$\frac{\partial^2 \Phi}{\partial u^2}(u,v) \cdot N(u,v) = a(u,v)\|N(u,v)\|^2 = a(u,v),$$

$$\frac{\partial^2 \Phi}{\partial u \partial v}(u,v) \cdot N(u,v) = b(u,v)\|N(u,v)\|^2 = b(u,v),$$

$$\frac{\partial^2 \Phi}{\partial v^2}(u,v) \cdot N(u,v) = c(u,v)\|N(u,v)\|^2 = c(u,v).$$

If we look at the definitions of the second fundamental form, we see that these equations imply that $a(u,v) = -e(u,v)$, $b(u,v) = -f(u,v)$, and $c(u,v) = -g(u,v)$. So the equations above with the Christoffel symbols become (I am going to omit writing the arguments of $E, F, G, e, f, g, \Gamma_{11}^1, \Gamma_{11}^2, \Gamma_{12}^1, \Gamma_{12}^2, \Gamma_{22}^1, \Gamma_{22}^2$ and the partials of Φ and N from now on in this section, they are all being evaluated at (u,v)):

$$\frac{\partial^2 \Phi}{\partial u^2} = \Gamma_{11}^1 \frac{\partial \Phi}{\partial u} + \Gamma_{11}^2 \frac{\partial \Phi}{\partial v} - eN,$$

$$\frac{\partial^2 \Phi}{\partial u \partial v} = \Gamma_{12}^1 \frac{\partial \Phi}{\partial u} + \Gamma_{12}^2 \frac{\partial \Phi}{\partial v} - fN,$$

$$\frac{\partial^2 \Phi}{\partial v^2} = \Gamma_{22}^1 \frac{\partial \Phi}{\partial u} + \Gamma_{22}^2 \frac{\partial \Phi}{\partial v} - gN.$$

It is in fact possible to solve for the Christoffel symbols in terms of the first fundamental form. To do that, dot both sides of the above equations by $\frac{\partial \Phi}{\partial u}$ to get that:

$$\frac{\partial^2 \Phi}{\partial u^2} \cdot \frac{\partial \Phi}{\partial u} = \Gamma_{11}^1 \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial u} + \Gamma_{11}^2 \frac{\partial \Phi}{\partial v} \cdot \frac{\partial \Phi}{\partial u},$$

$$\frac{\partial^2 \Phi}{\partial u \partial v} \cdot \frac{\partial \Phi}{\partial u} = \Gamma_{12}^1 \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial u} + \Gamma_{12}^2 \frac{\partial \Phi}{\partial v} \cdot \frac{\partial \Phi}{\partial u},$$

$$\frac{\partial^2 \Phi}{\partial v^2} \cdot \frac{\partial \Phi}{\partial u} = \Gamma_{22}^1 \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial u} + \Gamma_{22}^2 \frac{\partial \Phi}{\partial v} \cdot \frac{\partial \Phi}{\partial u}.$$

(the terms involving N went away since $N \perp \frac{\partial \Phi}{\partial u}$). Similarly, if you dot both sides of the previous equations by $\frac{\partial \Phi}{\partial v}$ you will get that:

$$\frac{\partial^2 \Phi}{\partial u^2} \cdot \frac{\partial \Phi}{\partial v} = \Gamma_{11}^1 \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v} + \Gamma_{11}^2 \frac{\partial \Phi}{\partial v} \cdot \frac{\partial \Phi}{\partial v},$$

$$\frac{\partial^2 \Phi}{\partial u \partial v} \cdot \frac{\partial \Phi}{\partial v} = \Gamma_{12}^1 \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v} + \Gamma_{12}^2 \frac{\partial \Phi}{\partial v} \cdot \frac{\partial \Phi}{\partial v},$$

$$\frac{\partial^2 \Phi}{\partial v^2} \cdot \frac{\partial \Phi}{\partial v} = \Gamma_{22}^1 \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v} + \Gamma_{22}^2 \frac{\partial \Phi}{\partial v} \cdot \frac{\partial \Phi}{\partial v}.$$

Combining the above two systems of equations and plugging in the definition of the first fundamental form gives that:

$$\frac{\partial^2 \Phi}{\partial u^2} \cdot \frac{\partial \Phi}{\partial u} = \Gamma_{11}^1 E + \Gamma_{11}^2 F,$$

$$\frac{\partial^2 \Phi}{\partial u \partial v} \cdot \frac{\partial \Phi}{\partial u} = \Gamma_{12}^1 E + \Gamma_{12}^2 F,$$

$$\frac{\partial^2\Phi}{\partial v^2}\cdot\frac{\partial\Phi}{\partial u} = \Gamma_{22}^1 E + \Gamma_{22}^2 F,$$

$$\frac{\partial^2\Phi}{\partial u^2}\cdot\frac{\partial\Phi}{\partial v} = \Gamma_{11}^1 F + \Gamma_{11}^2 G,$$

$$\frac{\partial^2\Phi}{\partial u\partial v}\cdot\frac{\partial\Phi}{\partial v} = \Gamma_{12}^1 F + \Gamma_{12}^2 G,$$

$$\frac{\partial^2\Phi}{\partial v^2}\cdot\frac{\partial\Phi}{\partial v} = \Gamma_{22}^1 F + \Gamma_{22}^2 G.$$

Let's express all of the quantities on the left-hand sides in terms of the first fundamental form. Differential integrating all of these quantities gives (I'm differential integrating them in a different order than listed above because differential integrating some uses the differential integration result of the other ones):

$$\frac{\partial^2\Phi}{\partial u^2}\cdot\frac{\partial\Phi}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\partial\Phi}{\partial u}\cdot\frac{\partial\Phi}{\partial u}\right) - \frac{\partial\Phi}{\partial u}\cdot\frac{\partial^2\Phi}{\partial u^2} \quad\Rightarrow\quad \frac{\partial^2\Phi}{\partial u^2}\cdot\frac{\partial\Phi}{\partial u} = \frac{1}{2}\frac{\partial}{\partial u}\left(\frac{\partial\Phi}{\partial u}\cdot\frac{\partial\Phi}{\partial u}\right) = \frac{1}{2}\frac{\partial E}{\partial u},$$

$$\frac{\partial^2\Phi}{\partial u\partial v}\cdot\frac{\partial\Phi}{\partial u} = \frac{\partial}{\partial v}\left(\frac{\partial\Phi}{\partial u}\cdot\frac{\partial\Phi}{\partial u}\right) - \frac{\partial^2\Phi}{\partial u}\cdot\frac{\partial^2\Phi}{\partial u\partial v} \quad\Rightarrow\quad \frac{\partial^2\Phi}{\partial u\partial v}\cdot\frac{\partial\Phi}{\partial u} = \frac{1}{2}\frac{\partial}{\partial v}\left(\frac{\partial\Phi}{\partial u}\cdot\frac{\partial\Phi}{\partial u}\right) = \frac{1}{2}\frac{\partial E}{\partial v},$$

$$\frac{\partial^2\Phi}{\partial u\partial v}\cdot\frac{\partial\Phi}{\partial v} = \frac{\partial}{\partial u}\left(\frac{\partial\Phi}{\partial v}\cdot\frac{\partial\Phi}{\partial v}\right) - \frac{\partial\Phi}{\partial v}\cdot\frac{\partial^2\Phi}{\partial v\partial u} \quad\Rightarrow\quad \frac{\partial^2\Phi}{\partial u\partial v}\cdot\frac{\partial\Phi}{\partial v} = \frac{1}{2}\frac{\partial}{\partial u}\left(\frac{\partial\Phi}{\partial v}\cdot\frac{\partial\Phi}{\partial v}\right) = \frac{1}{2}\frac{\partial G}{\partial u},$$

$$\frac{\partial^2\Phi}{\partial v^2}\cdot\frac{\partial\Phi}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\partial\Phi}{\partial v}\cdot\frac{\partial\Phi}{\partial v}\right) - \frac{\partial\Phi}{\partial v}\cdot\frac{\partial^2\Phi}{\partial v^2} \quad\Rightarrow\quad \frac{\partial^2\Phi}{\partial v^2}\cdot\frac{\partial\Phi}{\partial v} = \frac{1}{2}\frac{\partial}{\partial v}\left(\frac{\partial\Phi}{\partial v}\cdot\frac{\partial\Phi}{\partial v}\right) = \frac{1}{2}\frac{\partial G}{\partial v},$$

$$\frac{\partial^2\Phi}{\partial v^2}\cdot\frac{\partial\Phi}{\partial u} = \frac{\partial}{\partial v}\left(\frac{\partial\Phi}{\partial v}\cdot\frac{\partial\Phi}{\partial u}\right) - \frac{\partial\Phi}{\partial v}\cdot\frac{\partial^2\Phi}{\partial u\partial v} \quad\Rightarrow\quad \frac{\partial^2\Phi}{\partial v^2}\cdot\frac{\partial\Phi}{\partial u} = \frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u},$$

$$\frac{\partial^2\Phi}{\partial u^2}\cdot\frac{\partial\Phi}{\partial v} = \frac{\partial}{\partial u}\left(\frac{\partial\Phi}{\partial u}\cdot\frac{\partial\Phi}{\partial v}\right) - \frac{\partial\Phi}{\partial u}\cdot\frac{\partial^2\Phi}{\partial v\partial u} \quad\Rightarrow\quad \frac{\partial^2\Phi}{\partial u^2}\cdot\frac{\partial\Phi}{\partial v} = \frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v}.$$

Plugging these results into the previous system of equations gives us that:

$$\frac{1}{2}\frac{\partial E}{\partial u} = \Gamma_{11}^1 E + \Gamma_{11}^2 F,$$

$$\frac{1}{2}\frac{\partial E}{\partial v} = \Gamma_{12}^1 E + \Gamma_{12}^2 F,$$

$$\frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u} = \Gamma_{22}^1 E + \Gamma_{22}^2 F,$$

$$\frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v} = \Gamma_{11}^1 F + \Gamma_{11}^2 G,$$

$$\frac{1}{2}\frac{\partial G}{\partial u} = \Gamma_{12}^1 F + \Gamma_{12}^2 G,$$

$$\frac{1}{2}\frac{\partial G}{\partial v} = \Gamma_{22}^{1}F + \Gamma_{22}^{2}G.$$

This is a system of 6 linear equation for 6 unknowns. In fact, we can rewrite the above system of equations in matrix vector form:

$$\begin{bmatrix} \frac{1}{2}\frac{\partial E}{\partial u} \\ \frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma_{11}^{1} \\ \Gamma_{11}^{2} \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{2}\frac{\partial E}{\partial v} \\ \frac{1}{2}\frac{\partial G}{\partial u} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma_{12}^{1} \\ \Gamma_{12}^{2} \end{bmatrix},$$

$$\begin{bmatrix} \frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u} \\ \frac{1}{2}\frac{\partial G}{\partial v} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma_{22}^{1} \\ \Gamma_{22}^{2} \end{bmatrix}.$$

Multiplying both sides of the above equations by the inverse of $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ (which we by the way showed is always invertible in Section 4) we get that:

$$\begin{bmatrix} \Gamma_{11}^{1} \\ \Gamma_{11}^{2} \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} \frac{1}{2}\frac{\partial E}{\partial u} \\ \frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v} \end{bmatrix},$$

$$\begin{bmatrix} \Gamma_{12}^{1} \\ \Gamma_{12}^{2} \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} \frac{1}{2}\frac{\partial E}{\partial v} \\ \frac{1}{2}\frac{\partial G}{\partial u} \end{bmatrix},$$

$$\begin{bmatrix} \Gamma_{22}^{1} \\ \Gamma_{22}^{2} \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} \frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u} \\ \frac{1}{2}\frac{\partial G}{\partial v} \end{bmatrix}.$$

If you multiply the above equations out to get the equations for the Christoffel symbols, you will get that:

$$\Gamma_{11}^{1} = \frac{\frac{1}{2}\frac{\partial E}{\partial u}G - \frac{\partial F}{\partial u}F + \frac{1}{2}\frac{\partial E}{\partial v}F}{EG - F^2},$$

$$\Gamma_{11}^{2} = \frac{-\frac{1}{2}\frac{\partial E}{\partial u}F + \frac{\partial F}{\partial u}E - \frac{1}{2}\frac{\partial E}{\partial v}E}{EG - F^2},$$

$$\Gamma_{12}^1 = \frac{\frac{1}{2}\frac{\partial E}{\partial v}G - \frac{1}{2}\frac{\partial G}{\partial u}F}{EG - F^2},$$

$$\Gamma_{12}^2 = \frac{-\frac{1}{2}\frac{\partial E}{\partial v}F + \frac{1}{2}\frac{\partial G}{\partial u}E}{EG - F^2},$$

$$\Gamma_{22}^1 = \frac{\frac{\partial F}{\partial v}G - \frac{1}{2}\frac{\partial G}{\partial u}G - \frac{1}{2}\frac{\partial G}{\partial v}F}{EG - F^2},$$

$$\Gamma_{22}^2 = \frac{-\frac{\partial F}{\partial v}F + \frac{1}{2}\frac{\partial G}{\partial u}F + \frac{1}{2}\frac{\partial G}{\partial v}E}{EG - F^2}.$$

So the Christoffel symbols can be entirely expressed in terms of the first fundamental form (and its derivatives). Since the first fundamental form make up the entries of the metric tensor, we can restate this last statement as that the Christoffel symbols can be expressed entirely in terms of the entries of the metric tensor.

Section 7: Theorema Egregium

I want to include a section in this book on the theorem "Theorema Egregium" in memory of my teacher Professor Steve Mitchell who passed away on August 17, 2017. During my second year at the University of Washington I took topology and differential geometry with Professor Mitchell. Professor Mitchell was loved by all of his students as he inspired them with the enthusiasm and passion that he had for the subjects that he taught and was ready to help any student at any level. One of the admirable traits that Professor Mitchell portrayed while teaching this course was the incredible ease with which he moved and explained all of the topics covered in class. Indeed, the atmosphere in his classrooms was always of a mellow nature as he would try to convince us that everything we were doing was in fact simple and easy and was just a matter of "unraveling the definitions." His great and deep understanding of the subjects that he taught, combined with his friendly teaching style made him an outstanding instructor and mathematician. He had a great influence on my life and my perspective of differential geometry, topology, and mathematics in general.

One of my memories in Professor Mitchell's class was learning about the theorem "Theorema Egregium" (which translates to "Great Theorem" from Latin I believe). This theorem states that the Gaussian curvature is invariant under isometric maps. This is a fancy way of saying that if you have a map between two surfaces that preserves the local metric, then it preserves the Gaussian curvature. Folding a surface is an example of an isometric transformation and the example of folding a piece of paper into a cylinder shows that the analogous statement about the preservation of the mean curvature under isometric maps is false.

The way the theorem Theorema Egregium is proved is by showing that the Gaussian curvature can be expressed entirely in terms of the entries of the metric tensor. Indeed, metric preserving maps preserve the metric tensor because the metric tensor describes the metric of the surface locally. So if you show that the Gaussian curvature can be expressed entirely in terms of the entries of the metric tensor, then you will show that the Gaussian curvature is preserved under isometric maps. So in fact Theorema Egregium is a corollary of the following stronger theorem (due to Gauss).

Theorem 4.7.1 (Stronger Theorema Egregium): *The Gaussian curvature of a surface can be expressed entirely in terms of the entries of the metric tensor. The formula for the Gaussian curvature in terms of the entries of the metric tensor can be found at the end of this theorem's proof (below).*

We proved this theorem in Professor Mitchell's differential geometry class. I remember that after he showed us the proof of this theorem he said that, "you know, I've never seen such a beautiful theorem have such an ugly proof." Our proof of this theorem consisted of a very long calculation and he evidently considered such long calculations "ugly proofs."

Professor Mitchell told us that Gauss himself discovered the above theorem totally by accident while doing a calculation and was surprised to see that his surface curvature can be expressed entirely in terms of the first fundamental form (the entries of the metric tensor) and thus was invariant under isometries. During the summer between my second and third year as a student at the University of Washington I attempted to (unsuccessfully I believe) prove Theorema Egregium in higher dimensions. However in my attempt I discovered an absolutely elegant proof of the above Theorema Egregium that I am absolutely proud of. I never did get a chance to show Professor Mitchell the following beautiful proof to the above beautiful theorem, but I'm sure he would have loved it. A part of me believes that he heard my proof.

In the following I will use Newton's notation for partial derivatives.

Proof of the Stronger Theorema Egregium (Theorem 4.7.1): Let S be a smooth 2-dimensional surface sitting in $\mathbb{R}^3$ and $p \in S$ be any point on it. Since S is a smooth surface, there exists a surface parametrization Φ of S at p. Let $(u_0, v_0) = \Phi^{-1}(p)$. Now, let us calculate the following simple looking quantity (here I will omit the arguments of E, F, G, e, f, g, the partials of Φ, and the Christoffel symbols to make the equations shorter. They are being evaluated at (u_0, v_0)):

$$F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu}.$$

Plugging in the definition of the first fundamental form into the above quantity gives us that this quantity is equal to (here I will use the equality of mixed partials):

$$F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} = (\Phi_u \cdot \Phi_v)_{uv} - \frac{1}{2}(\Phi_u \cdot \Phi_u)_{vv} - \frac{1}{2}(\Phi_v \cdot \Phi_v)_{uu}$$

$$= \Phi_{uuv} \cdot \Phi_v + \Phi_{uu} \cdot \Phi_{vv} + \Phi_{uv} \cdot \Phi_{vu} + \Phi_u \cdot \Phi_{vuv} - \Phi_{uv} \cdot \Phi_{uv} - \Phi_u \cdot \Phi_{uvv} - \Phi_{vu} \cdot \Phi_{vu}$$

$$-\Phi_v \cdot \Phi_{vuu}$$

$$= \Phi_{uu} \cdot \Phi_{vv} - \Phi_{uv} \cdot \Phi_{uv}$$

It's amazing how many of the partials of Φ canceled out in the middle. Plugging in the definition of the Christoffel symbols into here gives us that:

$$F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu}$$

$$= (\Gamma_{11}^1\Phi_u + \Gamma_{11}^2\Phi_v - eN)\cdot(\Gamma_{22}^1\Phi_u + \Gamma_{22}^2\Phi_v - gN)$$

$$-(\Gamma_{12}^1\Phi_u + \Gamma_{12}^2\Phi_v - fN)\cdot(\Gamma_{12}^1\Phi_u + \Gamma_{12}^2\Phi_v - fN)$$

When I got to this point in the calculation during that summer I exclaimed "Eureka! I have discovered another proof of Theorema Egregium!" (we'll see in a moment why I exclaimed that). I remember that I couldn't control myself when I discovered this! I was so excited! Using the distributive property of the vector dot product in the above expression and the facts that $N \perp \Phi_u$ and $N \perp \Phi_v$, we get that:

$$F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu}$$

$$= \Gamma_{11}^1\Gamma_{22}^1E + \Gamma_{11}^1\Gamma_{22}^2F + \Gamma_{11}^2\Gamma_{22}^1F + \Gamma_{11}^2\Gamma_{22}^2G - (\Gamma_{12}^1)^2E - 2\Gamma_{12}^1\Gamma_{12}^2F - (\Gamma_{22}^2)^2G + eg - f^2$$

And after rearrangement we get that:

$$\frac{eg - f^2}{EG - F^2}$$

$$= \frac{1}{EG - F^2}\left(F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} - \Gamma_{11}^1\Gamma_{22}^1E - \Gamma_{11}^1\Gamma_{22}^2F - \Gamma_{11}^2\Gamma_{22}^1F - \Gamma_{11}^2\Gamma_{22}^2G + (\Gamma_{12}^1)^2E\right.$$

$$\left. + 2\Gamma_{12}^1\Gamma_{12}^2F + (\Gamma_{22}^2)^2G\right)$$

Since the Gaussian curvature K is equal to $(eg - f^2)/(EG - F^2)$ (see Definition 4.4.5), we get that the above equation implies that:

$$K = \frac{1}{EG - F^2}\left(F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} - \Gamma_{11}^1\Gamma_{22}^1E - \Gamma_{11}^1\Gamma_{22}^2F - \Gamma_{11}^2\Gamma_{22}^1F - \Gamma_{11}^2\Gamma_{22}^2G + (\Gamma_{12}^1)^2E\right.$$

$$\left. + 2\Gamma_{12}^1\Gamma_{12}^2F + (\Gamma_{12}^2)^2G\right)$$

So we were able to express the Gaussian curvature of K at $p \in S$ entirely in terms of the entries of the metric tensor and the Christoffel symbols. At the end of the previous section we showed that all of the Christoffel symbols can be expressed entirely in terms of the entries of the metric tensor. So the above equation implies that the Gaussian curvature can be expressed entirely in terms of the entries of the metric tensor.

■

As a point of interest, I would like to show you the nice and short form that the above formula for the Gaussian curvature takes when the surface parametrization is orthogonal. A surface parametrization $\Phi(u, v)$ of a surface S is called an **orthogonal surface parametrization** if for any (u, v) in the domain of Φ,

$$\frac{\partial\Phi}{\partial u}(u,v)\cdot\frac{\partial\Phi}{\partial v}(u,v)=0.$$

In other words, Φ is an orthogonal surface parametrization if the vectors $\frac{\partial\Phi}{\partial u}(u,v)$ and $\frac{\partial\Phi}{\partial v}(u,v)$ are constantly perpendicular to each other. Notice that the condition for a surface parametrization Φ to be orthogonal is equivalent to $F(u,v)$ being constantly zero over the domain of Φ. Orthogonal surface parametrizations are important in differential geometry because they always exist in a neighborhood of any point on a surface and in many situations the above property of a surface parametrization often makes equations and calculations shorter.

So, let us suppose that the surface parametrization Φ in the above equation for the Gaussian curvature is orthogonal. Let's see what the equation for K becomes. Plugging in the formulas for the Christoffel symbols in terms of the entries of the metric tensor that we derived at the end of the previous section into the above formula for the Gaussian curvature K and using the fact that $F(u,v)\equiv 0$ gives us that in the orthogonal surface parametrization Φ:

$$K=\frac{1}{EG}\left(-\frac{1}{2}E_{vv}-\frac{1}{2}G_{uu}+\frac{1}{(EG)^2}(\left(\frac{1}{2}GE_u\right)\left(\frac{1}{2}GG_u\right)E+\left(\frac{1}{2}EE_v\right)\left(\frac{1}{2}EG_v\right)G+\frac{1}{2}(GE_v)^2E\right.$$
$$\left.+\left(\frac{1}{2}EG_u\right)^2G\right).$$

I will leave it is as an exercise to the reader to show that this is equivalent to the famous formula:

$$K=-\frac{1}{2}\frac{1}{\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

in an orthogonal surface parametrization Φ. It's not hard to show that the two expressions above are equivalent, just carry out the partials in the above expression to see this. I hate it when authors do this sort of thing where they go from one expression to the other where the justification is just a reverse check. I've tried to avoid that throughout this book but in this case I can't do that because I honestly don't know how one goes from staring at the first expression above to arriving at the next. However, in any case we get the above beautiful expression for the Gaussian curvature in an orthogonal surface parametrization. I'm sure Professor Mitchell would have been proud!

Chapter 5: Variational Differential Geometry in $\mathbb{R}^3$

"That is the difference between mathematics and physics. Mathematicians, or people who have very mathematical minds, are often led astray when "studying" physics because they lose sight of the physics. They say: "Look, these differential equations—the Maxwell equations—are all there is to electrodynamics; it is admitted by the physicists that there is nothing which is not contained in the equations. The equations are complicated, but after all they are only mathematical equations and if I understand them mathematically inside out, I will understand the physics inside out." Only it doesn't work that way. Mathematicians who study physics with that point of view—and there have been many of them—usually make little contribution to physics and, in fact, little to mathematics. They fail because the actual physical situations in the real world are so complicated that it is necessary to have a much broader understanding of the equations.

What it means really to understand an [differential] equation—that is, in more than a strictly mathematical sense—was described by Dirac. He said: 'I understand what an equation means if I have a way of figuring out the characteristics of its solution without actually solving it.' So if we have a way of knowing what should happen in given circumstances without actually solving the equations, then we "understand" the equations, as applied to these circumstances." – Richard Feynman.[26]

Section 1: Outline (As Always)

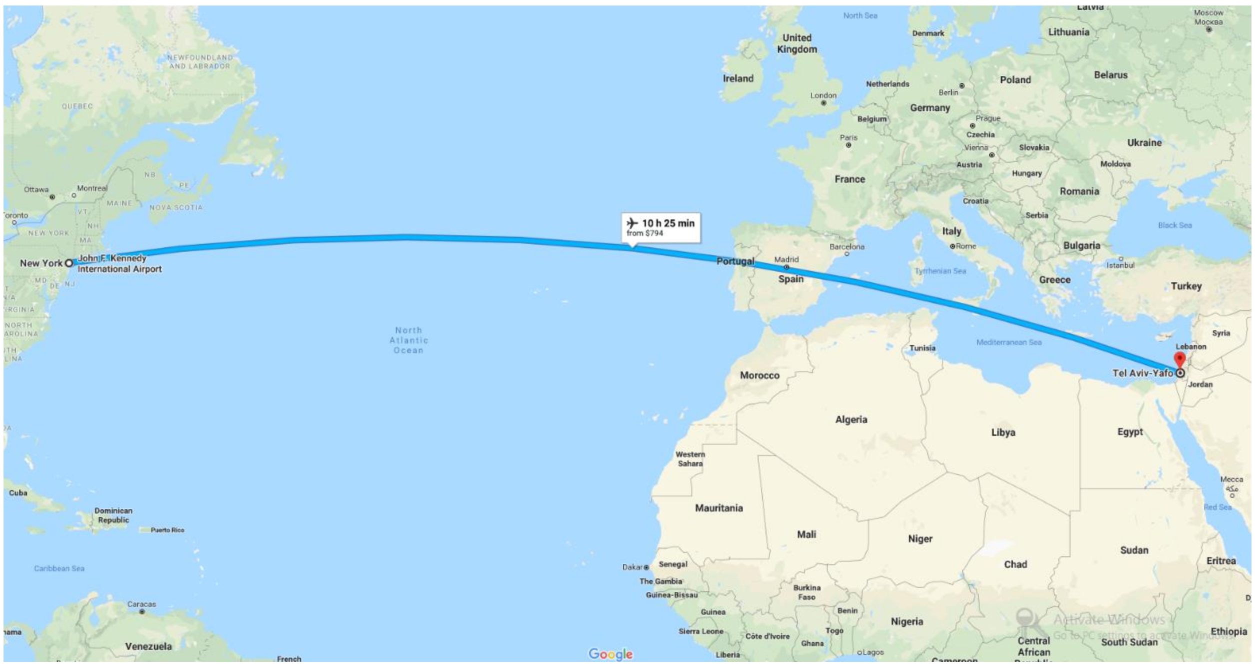
In this chapter we finally get to the exciting task of studying differential geometry from the perspective of the calculus of variations. Here we will prove three major theorems in differential geometry, two of which are naturally of variational nature. The three theorems are: the Minimizing Curve Theorem, the Minimal Surface Theorem, and a version of a corollary of the

[26] Reprinted with permission from the Feynman Lectures on Physics: http://www.feynmanlectures.caltech.edu/.

Global Gauss-Bonnet Theorem. In the next chapter we will repeat all of what we do here but on higher dimensional manifolds.

Section 2: String Theoretic Approach to Finding Minimizing curves[27]

One of my memories from my childhood is sitting on the flight between New York and Israel and looking at the slowly changing map in front of me that indicated how far along the flight we were. One of the interesting things that I noticed was that the flight path on the map was not a straight line but rather an arc that bended upwards.



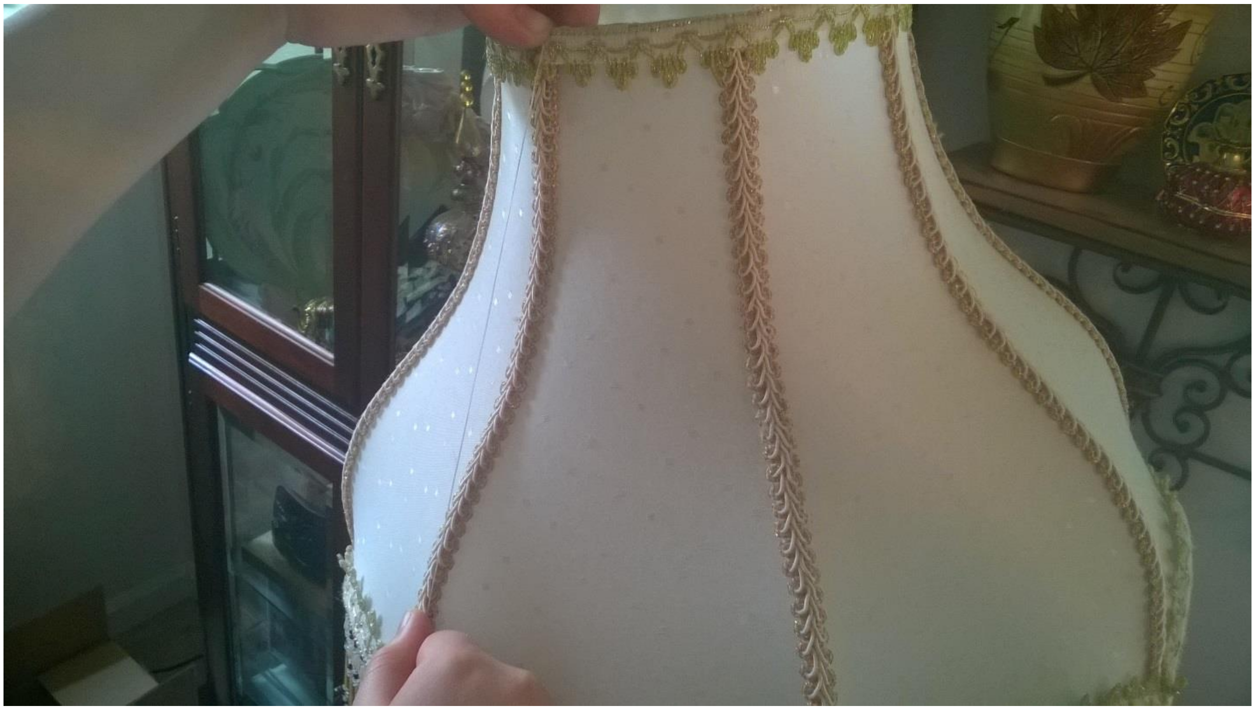
I asked my dad, “Papa, why is our flight path not a straight line but an arc like that. Isn’t the straight line the shortest route.” An understandable observation from a restless kid who was sitting on such a long flight. My dad answered, “It is true that in the two-dimensional plane the shortest path is a line. However, the shortest path on the map of the world does not actually represent the shortest path on the Earth because of the bent nature of the surface of our planet.” To explain his point, when we returned home from that trip my dad took a globe of the Earth and gave the following demonstration. He said, “Look, take a piece of string in both hands and put one finger on New York and the other on Israel. Now pull the string at both ends until you can no longer tighten it. You get that the string goes on some path between New York and Israel and this path will correspond to the shortest path on the globe between these two places. Notice that if you look at how this path looks like on the world map, it will not look like a straight line. This is why the straight lines on the world map do not indicate the shortest paths on the Earth. If you

[27] The purpose of the discussion in this section is solely to provide a physical intuition behind the minimizing curve equation. As a result, in the following discussion I will omit a certain level of rigor or precision. If you want, you can jump straight to the next section without loss of continuity.

ever want to find the shortest path between two places on the globe, you can do this with a piece of string as I've shown you." A demonstration that I would never forget!

It should be physically clear why the path that the string forms is the shortest path between two points on the sphere, at least locally to the path (in the curve norm). Indeed, if the path of the string does not represent the local arc-length minimum, then as you pull the string on both ends and take away the arc-length available for the string in the middle the forces acting on the string will shift (or "vary") the string path so as to accommodate the loss of available arc-length. Thus, when you can't tighten the string anymore the arc-length in the middle will be a local minimum.

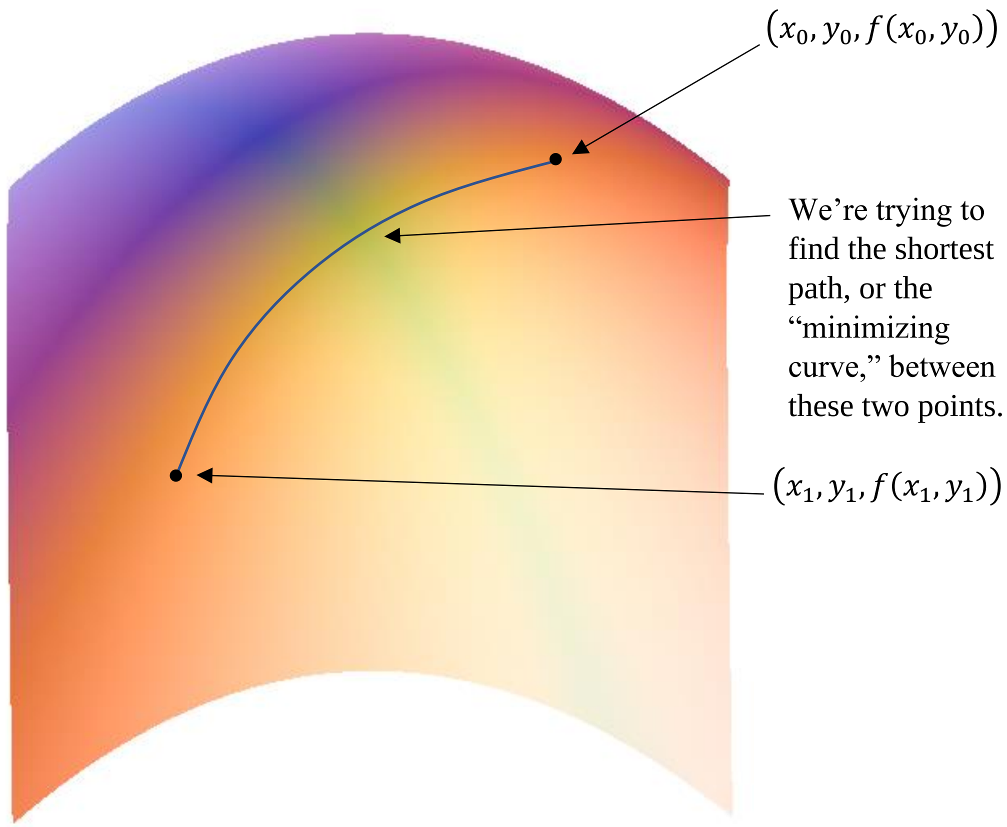
Curves that locally minimize arc-length are called **arclength minimizing curves** or just simply **minimizing curves**. From a young age I realized that this trick with a string to find minimizing curves on the sphere works on other surfaces as well. As a young kid I would walk around the house with a piece of string and find minimizing curves on all sorts of surfaces that I could find in our house. This trick of finding the shortest path between two points on a surface even works on non-smooth surfaces such as two points on the opposite sides of a sharp edge of a shelf. However, this trick of finding the minimizing curve between two points on a surface obviously has some limitations since if you're on one side of a surface that's curving towards you, then the tightened string path between your two points will always be a line and not the surface's minimizing curve (it will be the minimizing curve in three-dimensional space!).



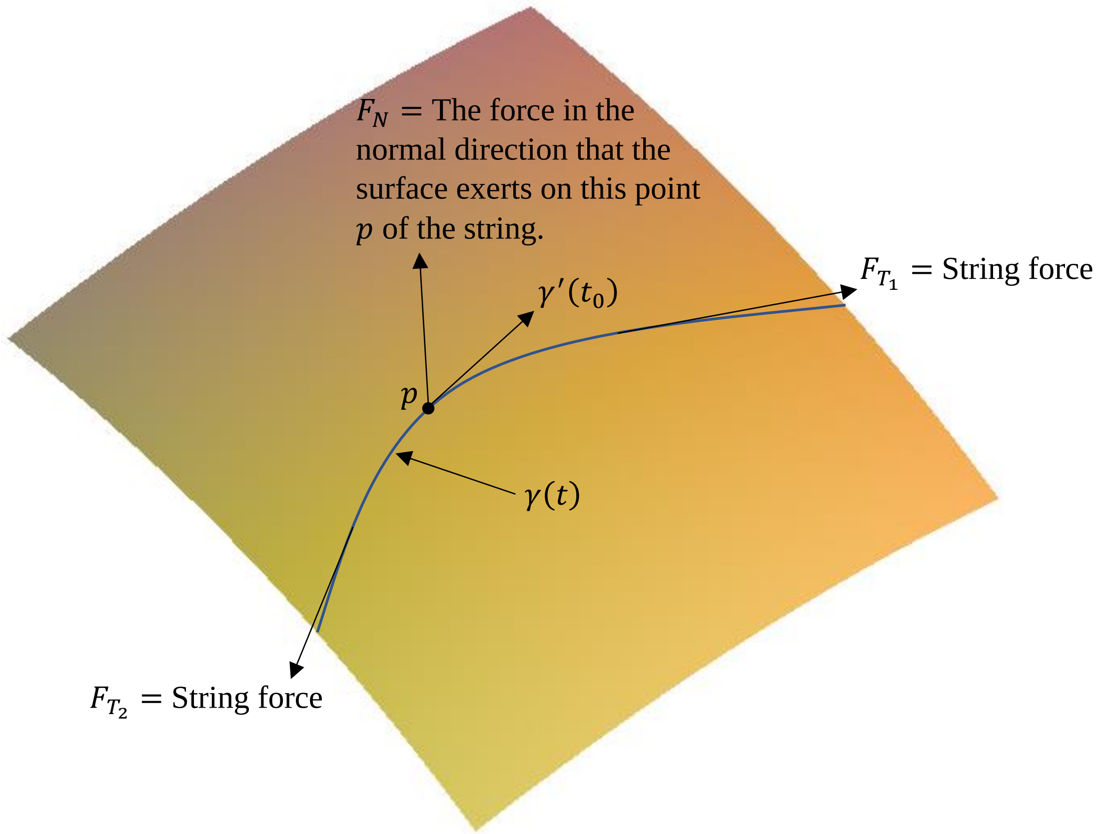
But on most surfaces if your standing on the right side of the surface, locally enough to any point this trick always works.

The problem of finding minimizing curves between two points on a surface fascinated me ever since my dad showed me the above demonstration with a string and a globe. At one point during my first year at the University of Washington I decided to sit down and try to write out an equation for the minimizing curve between two points on an arbitrary surface. I knew that finding an explicit equation for the minimizing curve on an arbitrary surface is probably impossible, but what I can do is try to find the next best thing: a differential equation for the minimizing curve.

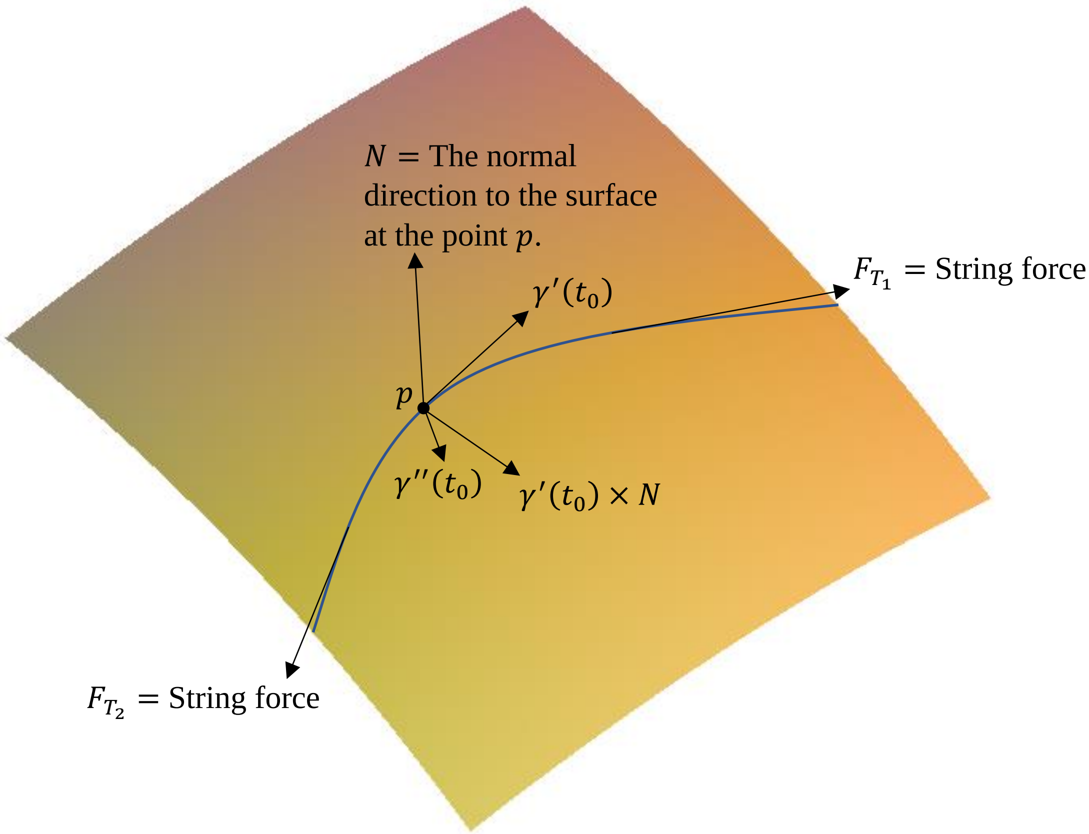
I thought, "Let's suppose that we have a surface that is the graph of the real-valued function $f(x, y)$. Now, take two distinct points $(x_0, y_0, f(x_0, y_0))$ and $(x_1, y_1, f(x_1, y_1))$ on the surface. Let's try to find the minimizing curve between them (I was having a conversation with my conscience)."



I didn't know how to do this – I didn't know any calculus of variations at the time. So I decided to try to find the minimizing curve by the trick that my dad showed me as a kid using a piece of string. I imagined taking a piece of string and stretching it between the two points on the surface. Let $\gamma(t)$ denote the curve that represents the path that the string will lie on. Let's suppose that we can parametrize this path so that $\gamma'(t)$ is never zero (this is done so as to give some geometric regularity to the curve). When the string is fully stretched, it will be in a physically stable position. This means the if we draw the force diagram at each point of the string, all of the forces have to cancel out. The force diagram at any point p on the string is given by (here t_0 is the time when $\gamma(t)$ passes through p):



I then thought: "Ok, if the string forces, which arise from the tension on the string, exert any sideways forces on the differential piece of string at the point p, then the string would start shifting (or "sliding") in that sideways direction on the surface. Physically one can see that the string forces will exert a sideways force on the differential piece of string at the point p if the curve $\gamma(t)$ is curving sideways on the surface in relation to the tangential vector $\gamma'(t_0)$. But since our string path is stationary, this cannot happen. Now, the direction that is tangent to the surface and that is perpendicular to the path of the string at p is given by: $\gamma'(t_0) \times N$ where N is a unit normal direction to the surface at the point p."



I continued to think: "So in order for the string path not to curve sideways on the surface in relation to the tangential vector $\gamma'(t_0)$ (in order not to start sliding sideways), we must have that the component of the second derivative of $\gamma(t)$ at p in the direction $\gamma'(t_0) \times N$ is zero. Thus, the path of the string must satisfy the following equation at p:

$$\gamma''(t_0) \cdot (\gamma'(t_0) \times N) = 0.$$

Since p can be any point on the string's path, I concluded that the string path must satisfy the following differential equation:

$$\gamma''(t) \cdot \big(\gamma'(t) \times N(t)\big)$$

for all times t where $N(t)$ denotes the unit normal vector to the surface at the point $\gamma(t)$." Since the string path is supposed to represent the shortest path on a surface between the two points, I hypothesized that the above equation was the differential equation that all minimizing curves must satisfy. It was guess back then and I wasn't sure whether it was right or wrong. If it was right, then this would have been an amazing equation because then if I wanted to find the shortest path between two points on a surface, I just had to solve the above differential equation for $\gamma(t)$ that satisfied the boundary conditions of passing through my two points.

I did write the above equation in the above form at the time, but I also wrote it in a particular parametrization. At the time, my favorite way to parametrize any surface was to set it to be the graph of a function $f(x, y)$ and any surface curve on it as:

$$\gamma(x) = \left(x, y(x), f\left(x, y(x)\right)\right)$$

where $y(x)$ is some function of x. Then, if you plug this in into the above vector differential equation you will get that in order to find the $y(x)$ that parametrizes the surface minimizing curve, you have to solve the following differential equation for $y(x)$ (here all of the partials of f are being evaluated at $\left(x, y(x)\right)$):

$$\left(-\frac{\partial f}{\partial x}\cdot\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'\right)-1\right)y''+\left(\frac{\partial f}{\partial x}y'-\frac{\partial f}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\partial y}y'+\frac{\partial^2 f}{\partial y^2}{y'}^2+\frac{\partial f}{\partial y}y''\right)=0.$$

A scary equation, but the excitement that it stirred in me was beyond measure!

Back then I was very interested in the problem of finding minimizing curves on the paraboloid (the surface given by the graph of the function $f(x, y) = x^2 + y^2$). By that point I found the minimizing curves in the plane and the sphere and so the paraboloid was a natural choice for a surface to look for minimizing curves on next. If you plug in $x^2 + y^2$ into $f(x, y)$ in the above equation, you will get that the above differential equation becomes (after a little bit of rearrangement):

$$-y''(4x^2+4y^2+1)+4xy'-4y{y'}^2+4x{y'}^3-4y=0.$$

I cannot solve this differential equation, but it's cool to consider that this differential equation describes the minimizing curves on the paraboloid. You can always for example try to tackle this differential equation using numerical means. I remember that I once showed the above equation and the above vector differential equation to my first-year calculus teacher as my hypotheses for differential equations for minimizing curves. After trying his hands for a few minutes at solving the above differential equation, he couldn't tell me anything more including whether or not my hypotheses were correct (which is completely understandable since this problem of finding minimizing curves is extremely difficult).

My certainty of whether the above vector differential equation was indeed the differential equation for the minimizing curves on surfaces had its extreme ups and downs, ranging from being 100 percent that I found the correct differential equation to being 95 percent sure that I was wrong. It took me a whole 1.5 years before I finally figured out that the above differential equation for minimizing curves is in fact correct.

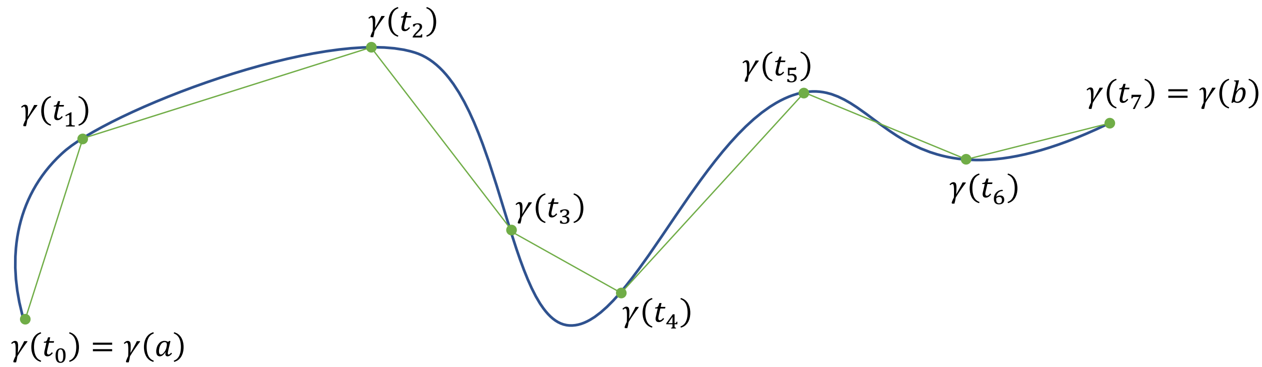
I would like to end this section with a comment on the above vector differential equation. It is a shame sometimes that in many disciplines the solutions to problems are often differential equations and not explicit forms. For example, this is the case in the calculus of variations where the solution is always the Euler-Lagrange differential equation. In such fields then, the ability to make deductions often depend on either your ability to solve differential equations or your ability to describe the nature of solutions to differential equations – two things that are often very difficult. We will prove in the fourth section that the above vector differential equation is indeed the differential equation for the minimizing curves and it turns out that in practical applications it's almost never possible to solve this differential equation explicitly. However, the above differential equation does have the nice property that it carries behind it a physical statement. It basically says that minimizing curves can never curve away from the direction that they are going in. Indeed this is understandable since if you zoom in greatly on any portion of the curve, the surface will look pretty much flat (just like the Earth seems pretty much flat to humans) and if we would see some curving sideways of the curve along the surface, then it's geometrically clear that we could straighten out the path a little bit in order to get a shorter curve. For this reason, minimizing curves cannot curve away from the direction that they are going in and mathematically we write this down as $\gamma''(t) \cdot \left(\gamma'(t) \times N(t)\right) \equiv 0$ ($\gamma''(t)$ describes the "curving" nature of the curve). Another way to interpret this equation is that $\gamma''(t)$ must constantly lie in the plane spanned by $\gamma'(t)$ and $N(t)$. In other words, minimizing curves can only accelerate in the direction of motion and normal direction to the surface.

In some situations, it may be possible to solve the above minimizing curves equation or at least tell the nature of its solution by considering the geometric nature of the surface itself. Such a study will for example be the subject of the next section and a section in Chapter 8 [see future edition of this book].

Section 3: Minimizing curves on the Plane, Sphere, and Surfaces of Revolution

An experience that probably all of us went through in our lifetimes at some point is hearing from our geometry teacher that the straight line is the shortest path between any two points on the plane or in three-dimensional space and then rolling our eyes thinking "well, that's obvious." Indeed it is intuitively obvious and drawing on the ideas from the previous section one amusing way to check this is by taking two points in space and tightening a string between them. Indeed you will get that the shortest path is the straight line! A rigorous proof of the fact that the straight line is the shortest path on the plane really depends on the starting point of your definition of "arclength." There are two main definitions of arclength, one of which defines arclength for a large class of curves called "rectifiable curves" and the other defines it for all continuously differentiable curves. Let's take a look at them.

The first definition of arclength of a curve comes from the natural idea of approximating the arclength of a curve by polygonal approximations. For example, if we take some curve $\gamma : [a, b] \to \mathbb{R}^2$ in the plane, we can partition up the domain interval $[a, b]$ into many subdivisions $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ and then consider the polygonal path that they create (in the following picture, $n = 7$):



Now, what is the length of the polygonal approximation to the curve? It is the sum of the lengths of all the individual line segments:

$$\|\gamma(t_1) - \gamma(t_0)\| + \|\gamma(t_2) - \gamma(t_1)\| + \cdots + \|\gamma(t_n) - \gamma(t_{n-1})\| = \sum_{k=0}^{n-1} \|\gamma(t_{k+1}) - \gamma(t_k)\|.$$

Then this is an approximation to the arclength of the curve $\gamma(t)$. If we partition up our domain interval even more, this approximation will get better and better. One way to define the arclength of the curve $\gamma(t)$ from here is now to set up a limit. That is a very nice way to define the arclength and it is connected to the second definition of arclength that we will give below. However in this case, since each of the polygonal paths under approximate the length of the actual curve, mathematicians simply defined the arclength of the curve $\gamma(t)$ to be the supremum of all such approximations. This is our first definition of arclength of a curve.

Definition 5.3.1 (First Definition of Arclength): *Let $\gamma : [a, b] \to \mathbb{R}^2$ be an injective curve in the plane. Let $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ be the any finite partition of the interval $[a, b]$:*

$$t_0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = t_1.$$

Now consider the sum:

$$L_{\mathcal{P}}[\gamma(t)] = \sum_{k=0}^{n-1} \|\gamma(t_{k+1}) - \gamma(t_k)\|.$$

Another way to write this sum is:

$$L_{\mathcal{P}}[\gamma(t)] = \sum_{\mathcal{P}} \|\gamma(t_{k+1}) - \gamma(t_k)\|.$$

*Now, if the supremum of all such sums is finite, then this supremum is defined as the **arclength** of the curve $\gamma(t)$. If this supremum exists, then the curve $\gamma(t)$ is called **rectifiable**. If we let $\mathcal{P}[a, b]$ denote the set of all finite partitions of the interval $[a, b]$, then the arclength of $\gamma(t)$ can be neatly written as:*

$$L[\gamma(t)] = \sup_{\mathcal{P} \in \mathcal{P}[a,b]} \{L_{\mathcal{P}}[\gamma(t)]\} = \sup_{\mathcal{P} \in \mathcal{P}[a,b]} \left\{\sum_{\mathcal{P}} \|\gamma(t_{k+1}) - \gamma(t_k)\|\right\}.$$

The other way to define arclength should be familiar from calculus. In calculus, you were probably given the following definition of arclength.

Definition 5.3.2 (Second Definition of Arclength): *Let $\gamma : [a,b] \to \mathbb{R}^2$ be an injective $C^2[a,b]$ curve in the plane. Then the arclength of the curve $\gamma(t)$ is given by:*

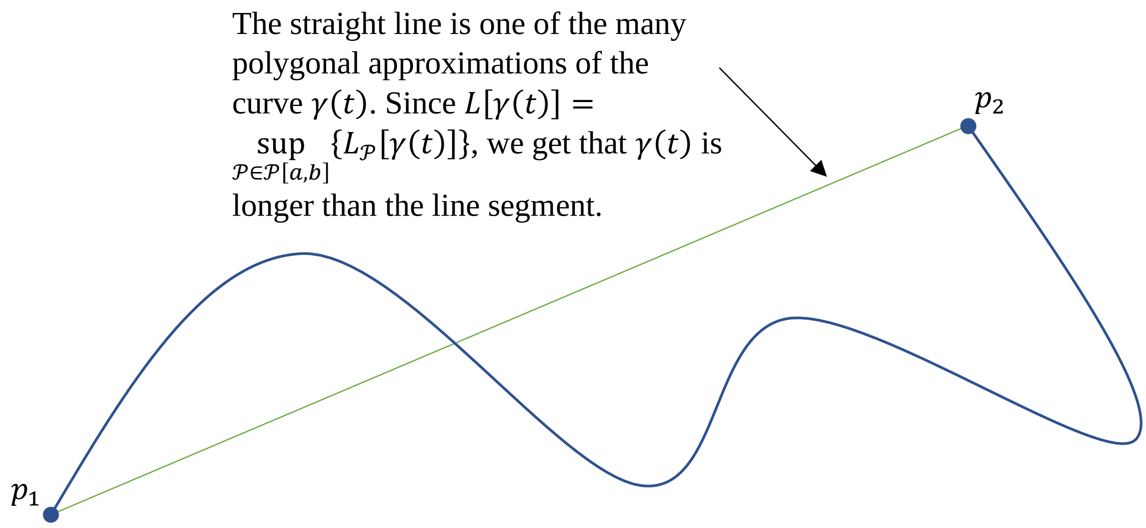
$$L[\gamma(t)] = \int_a^b \|\gamma'(t)\| dt.$$

It turns out that the second definition is a special case of the first definition in the sense that the arclength of any curve $\gamma \in C^2[a,b]$ in the plane is the same according to both definition. This fact is a standard theorem that is often proved in a calculus course and so we will not prove it here. It should also be pointed out that the above two definitions of arclength are well defined in terms of reparameterizations. Indeed, in the case of the second definition this is just given by the change of variables formula for integrals: if $\gamma \in C^2[a,b]$ is a curve in the plane and $f(s)$ is a reparameterization function of the form $f : [c,d] \to \mathbb{R}$ (being a reparameterization function means that it's strictly increasing and continuously differentiable), then $\gamma\big(f(s)\big)$ is a reparameterization of the curve $\gamma(t)$ and their arclengths according to Definition 5.3.2 are the same (here I do a change of variables in the integral):

$$L[\gamma(t)] = \int_a^b \|\gamma'(t)\| dt = \int_c^d \big\|\gamma'\big(f(s)\big)\big\| f'(s) ds = \int_c^d \frac{d}{ds}\Big(\big\|\gamma\big(f(s)\big)\big\|\Big) ds = L\big[\gamma\big(f(s)\big)\big].$$

So Definition 5.3.2 is well defined in terms of reparameterizations of a curve. The verification that Definition 5.3.1 is also well defined in terms of reparameterizations isn't hard and I will leave it to the reader. It should also be noted that if p_1 and p_2 are two distinct points in the plane, then the arclength of the line segment between these two points is equal to $\|p_2 - p_1\|$ by both of the above definitions of arclength. The verification of this is an easy exercise that I will leave this to the reader as well.

Starting from either definition of arclength, it is pretty quick to see that the lines are shortest paths in the plane. Indeed, take any two distinct points on the plane p_1 and p_2 and any curve $\gamma : [a,b] \to \mathbb{R}^2$ that goes between them (meaning that $\gamma(a) = p_1$ and $\gamma(b) = p_2$). Notice that the straight line between them is one of the many possible polygonal approximations of the curve $\gamma(t)$ and so since the arclength of $\gamma(t)$ is the supremum of the length of polygonal approximations to the curve, we get that the arclength of $\gamma(t)$ is by definition longer than or equal to the arclength of the straight line between them.



So if the first definition of arclength is our starting point, then it is almost immediate the lines in the plane are shortest paths. I only say that they are shortest paths and not *the* shortest paths in the plane because the above only proves that they are shorter or equal in length to any other curve in the plane between any two points. But there might be other such curves in the plane. So the above argument does not give uniqueness to the minimizing curve in the plane. It does however turn out, as you probably suspected, that the lines are the unique minimizing curves between two points in the plane and the argument with circles that we will give below will prove this uniqueness portion of this argument.

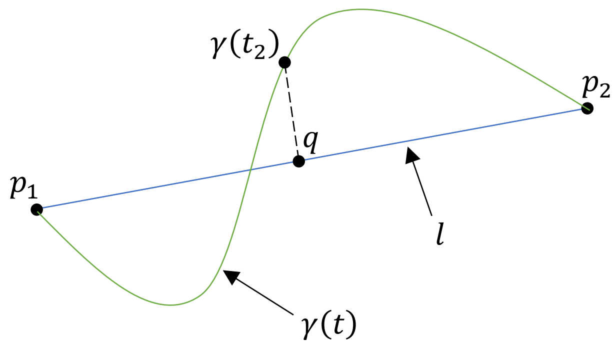
Now, what if the second definition of arclength is our starting point? If the second definition of arclength is our starting point, then a similar argument like above is possible. However, there is a much easier way to obtain the same arclength inequality by using the triangle inequality for integrals. Let's take the curve $\gamma(t)$ from above but let's add the assumption that $\gamma \in C^2[a,b]$. Notice that we have that:

$$\|p_1 - p_2\| = \|\gamma(b) - \gamma(a)\| = \left\| \int_a^b \gamma'(t)dt \right\| \leq \int_a^b \|\gamma'(t)\|dt = L[\gamma(t)].$$

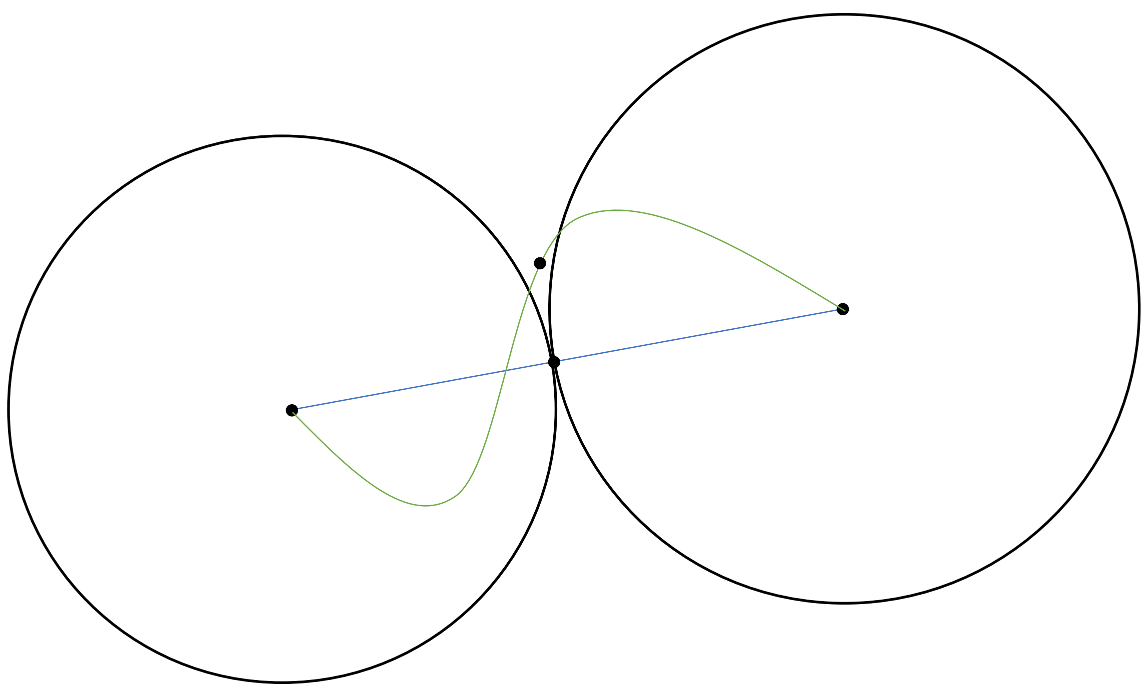
Since the length of the line segment is $\|p_1 - p_2\|$, we get that the arclength of the curve $\gamma(t)$ is longer than or equal to the length of the line segment. So with the second definition we also get that the lines are shortest paths in the plane.

The above proofs of the fact that the straight lines are shortest paths in the plane might feel a little bit unsatisfactory because in both cases the result comes out almost by definition. As already mentioned, the second definition is nontrivially a special case of the first definition. And the idea, or as some people say "motivation," behind the first definition already draws upon the fact that the line segments are the shortest path in the plane. We drew upon this fact when we said the phrase "each of the polygonal paths under approximates the length of the actual curve" right before stating Definition 5.3.1. Although the above are rigorous proofs of the fact that the straight lines are shortest paths in the plane from the perspective of the above definitions of arclength, it would be nice to see if there is a simple geometric reason behind why the straight lines are the shortest paths in the plane.

The following is a qualitive and geometric reason behind why the straight lines are the only candidates for the minimizing curves in the plane.[28] Let us take any two distinct points in the plane and call them p_1 and p_2. Suppose that there was some curve $\gamma : [t_0, t_1] \to \mathbb{R}^2$ between p_1 and p_2 other than the straight-line segment between them that is shorter than or equal to any curve between p_1 and p_2. Let l denote the line segment between p_1 and p_2. Since $\gamma(t)$ is not the straight line, we can find a time $t_2 \in [t_0, t_1]$ such that $\gamma(t_2)$ is not on the line segment l between p_1 and p_2. Now, take the projection q of $\gamma(t_2)$ onto the line segment l like so:

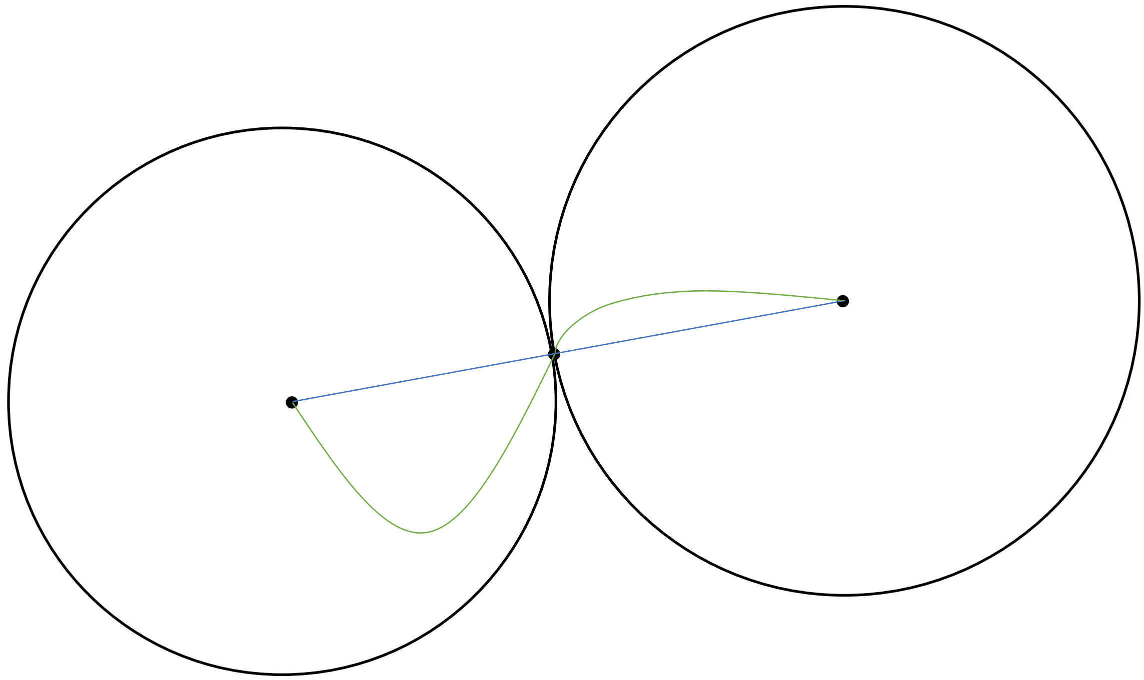


Draw two circles centered at the points p_1 and p_2 that go through q.



[28] I thought of the following argument while sitting in the car of a family car trip. What's better to do on a family car trip than to daydream in math equations and diagrams!

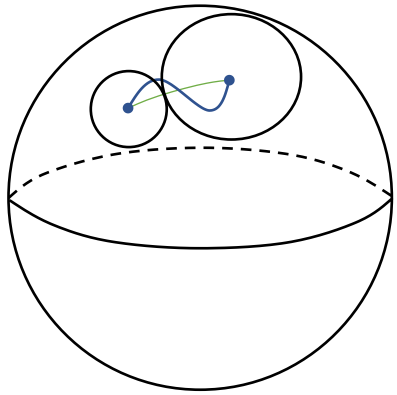
Notice that we can now throw away the portion of the curve that is not in the circles and take the portions of the curves that are in each circle and rotate them so that the meet at q.



Notice that with this we were able to construct a shorter path between p_1 and p_2 because we threw out a middle portion of the curve. But this contradicts the fact that $\gamma(t)$ is a minimizing curve (or "shortest path") between p_1 and p_2 since $\gamma(t)$ being a minimizing curve means that no shorter path between p_1 and p_2 should exist. We have a contradiction and thus $\gamma(t)$ cannot be a minimizing curve. Thus this argument shows that the straight lines are the only candidates for the minimizing curves in the plane. In fact, since we proved that the lines are minimizing curves between points in the plane, the above argument gives the uniqueness to the minimizing curves in the plane. Thus the line segment between any two points is the unique shortest path.

I do want to warn you that the above argument with the circles alone doesn't prove that the straight lines are the minimizing curves in the plane. Remember, in order to show that the straight lines are minimizing curves in the plane you have to show that every other curve that connects your two points is longer or equal to in length to the line segment. The above argument however only shows that curves that are not line segments cannot be minimizing curves. But the above doesn't show that all non-line-segment curves are longer than or equal to in length to the line segments. It turns out that it's not too difficult to modify the above circles argument to actually prove that the line segments are the minimizing curves in the plane. However, any attempt to do so will inevitably lead you back to Definitions 5.3.1 and 5.3.2 of the arclength of a curve and so one might as well just use the simpler proofs presented earlier in this section of this fact.

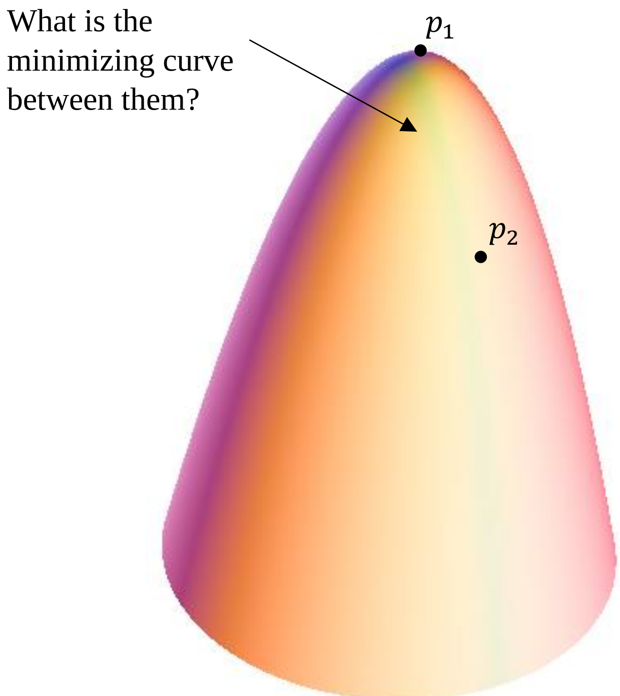
The above technique with the circles however has much more value when you apply it to the sphere where the question of what are the minimizing curves is much less trivial. Indeed, you can use the above technique to show that the only candidates for the minimizing curves on the sphere are the great arcs (these are arcs obtained by intersecting planes going through the origin and the sphere). One of the diagrams in the middle of your argument will look something like:



The two circles in the above picture are obtained by taking spheres centered at each of the endpoints and then taking their intersection with the surface. In addition, it is possible to give a separate argument that proves that the great arcs are actual minimizing curves on the sphere which coupled with the above argument will give that the great arcs are the unique minimizing curves between two points on a sphere.[29]

The whole reason why the above arguments with the circles work on the plane and the sphere is that both the plane and the sphere are spaces that possess a very nice symmetric property. Both the sphere and the plane are radially symmetric around any point. I believe that no other surface has this similar sort of symmetry property, but this doesn't mean that this circles' trick cannot be applied elsewhere. Some surfaces may not possess such a symmetry property around every point, but they may posses it at one point in which case this trick can be applied in a one-sided fashion. To illustrate this point, let us take a surface of revolution S that is obtained by rotating a function of the form $z = f(x)$ around the z-axis. Let p_1 be the point $\big(0,0,f(0)\big)$ and p_2 be any other point on the surface. We can now ask the question: what is the minimizing curve from the point p_1 to p_2?

[29] Care has to be taken with this statement since great arcs can go the long way around the sphere and the short way between the two points. The minimizing curves are always the short way between the two points along a great arc.



Notice that p_1 has the mentioned symmetry property in relation to the surface. Indeed, the surface S is radially symmetric around p_1. It turns out that the answer to the question of what is the minimizing curve between p_1 and p_2 is rather intuitive. In the context of the above surface of revolution, which looks like a mountain, you just have start at p_1 and descend straight downwards toward p_2 in order to walk upon the shortest path. In mathematical terms, if we write:

$$p_1 = \left(p_{1_x}, p_{1_y}, p_{1_z}\right) = \left(0,0, f(0)\right),$$

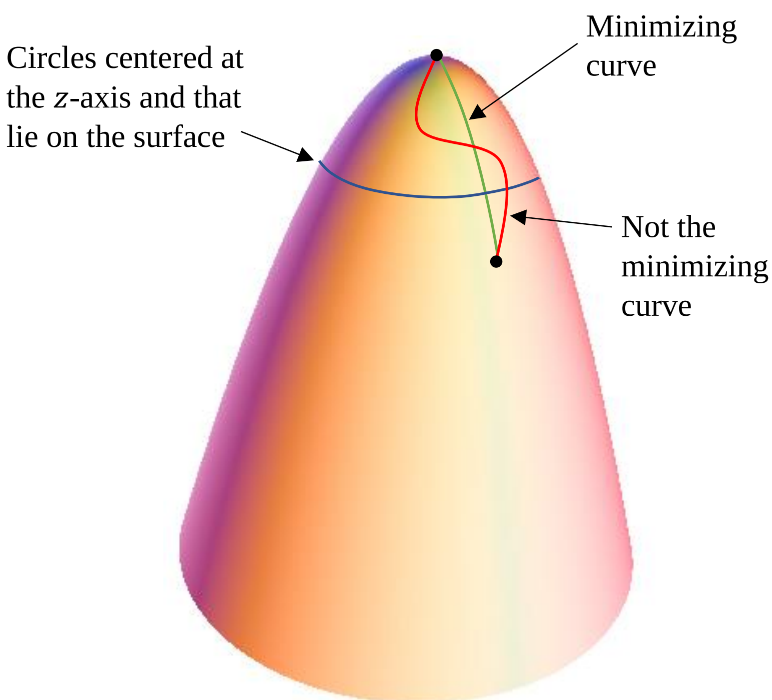
$$p_2 = \left(p_{2_x}, p_{2_y}, p_{2_z}\right).$$

Then the minimizing curve between p_1 and p_2 is given by the curve $\gamma : [0, 1] \to \mathbb{R}^3$ defined by:[30]

$$\gamma(t) = \left(p_{2_x}t, p_{2_y}t, f\left(\sqrt{p_{2_x}^2 + p_{2_y}^2}\,t\right)\right)$$

on any such surface of revolution. And the way this is proven is by exploiting the symmetry of the surface S around the point p_1. You have to start by drawing circles centered at the z-axis and that lie on the surface. From there, using these circles like before it is possible to prove that the above curve is indeed the minimizing curve between p_1 and p_2.

[30] This is only one of the many possible parametrizations for this curve.



The argument is a bit tricky but not too difficult. In fact, since this argument is of a very geometric nature it is possible to use this argument on more general surfaces of revolution that are not simply generated by taking a function of the form $z = f(x)$ and rotating it around the z-axis. Surfaces of revolution that this argument works on for example include the sphere and so this type of argument furnishes a nice proof of the fact that the great arcs are the minimizing curves on the sphere. In Chapter 8 [see future edition of this book] we will talk about another geometric way of finding minimizing curves, a method I like to call "the method of secants."

Section 4: The Minimizing Curve Theorem

In this section we finally get to our first variational theorem in differential geometry: The Minimizing Curve Theorem. This theorem characterizes a necessary condition that all minimizing curves on surfaces must satisfy and thus provides a good way of finding them in practice. Before we get to the theorem, let's give a precise mathematical definition of a minimizing curve.

Definition 5.4.1: *Suppose that S is a smooth 2-dimensional surface sitting in $\mathbb{R}^3$ and let p_1 and p_2 be any two distinct points on S. Then a curve $\gamma \in C^2[t_0, t_1]$ in $\mathbb{R}^3$ that lies on the surface is called a **minimizing curve** between p_1 and p_2 if it connects the two points:*

$$\gamma(t_0) = p_1 \quad and \quad \gamma(t_1) = p_2,$$

its derivative is nonvanishing, and for any other surface curve $\eta \in C^2[t_0, t_1]$ that connects these two points and whose derivative never vanishes, the arclength of $\gamma(t)$ is less than or equal to the arclength of $\eta(t)$:

$$L[\gamma(t)] \leq L[\eta(t)].$$

In other words, a minimizing curve between two points on a surface is a surface path of shortest arclength. The reason why we require that the derivatives of the curves involved don't vanish is to prevent the image of the curves from having awkward corners. In most situations the minimizing curve is unique up to reparameterization, but this isn't always the case. For example, if our surface is the sphere and out two points are the north and south pole respectively, then by the symmetry of the surface it clear that the minimizing curve between these two points will not be unique, even up to reparameterization.

Notice that the problem of finding minimizing curves between two points is naturally a variational one. In this case, finding the minimizing curve between two points is equivalent to finding the minimum of the functional that returns the arclength of any curve that goes between your two points. So using the Euler-Lagrange differential equation to try to find minimizing curves on surfaces is the most natural approach to this problem. So let us do that in the following theorem.

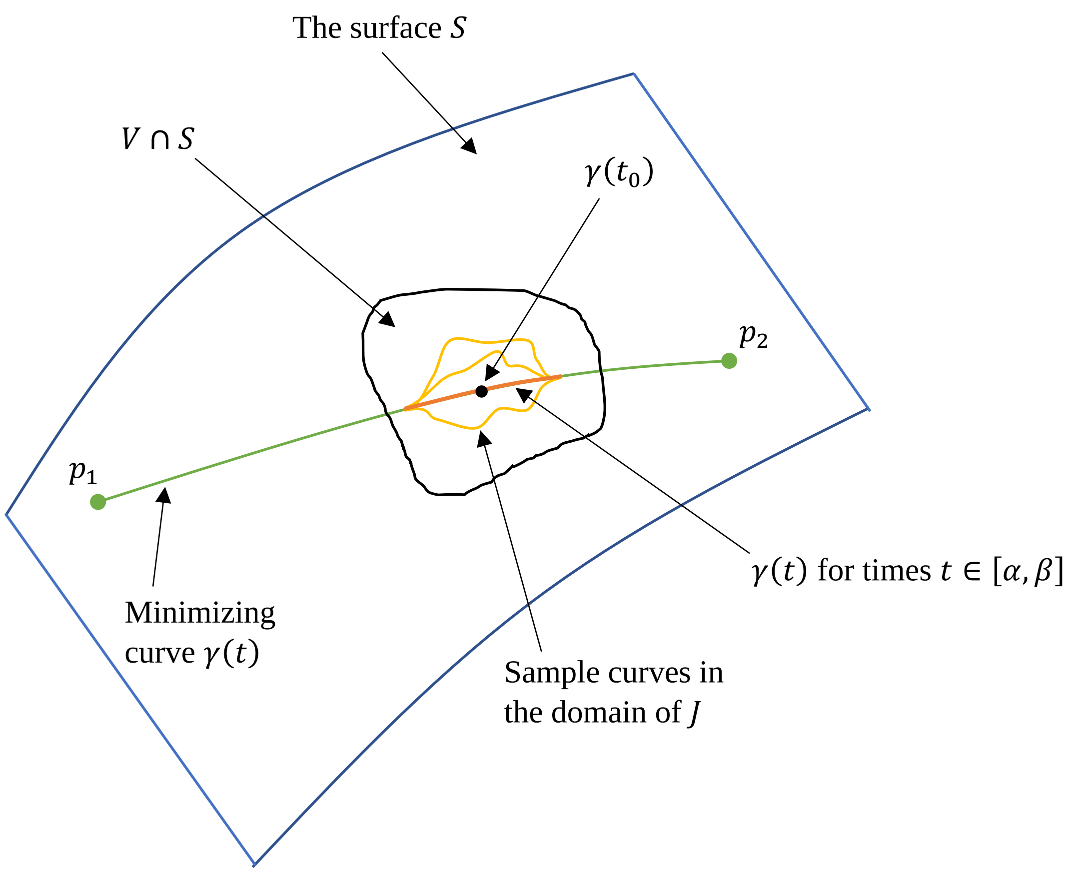
The main idea in the following proof came from something that Gelfand and Fomin presented in their calculus of variations book.

Theorem 5.4.2 (The Minimizing Curve Theorem): *Suppose that S is a smooth 2-dimensional surface sitting in $\mathbb{R}^3$ and let p_1 and p_2 be any two distinct points on S. Suppose that $\gamma \in C^2[a,b]$ is a minimizing curve between p_1 and p_2. Then for any time $t \in (a,b)$, our minimizing curve satisfies the* ***minimizing curve equation****:*

$$\gamma''(t) \cdot (\gamma'(t) \times N) = 0$$

where N is any nonzero normal vector to the surface at the point $\gamma(t)$.

Proof: To prove this theorem we will use the surface Euler-Lagrange vector differential equation that we derived in Chapter 3 (Theorem 3.3.1). Let us take any time $t_0 \in (a,b)$ and show that the minimizing curve satisfies the minimizing curve equation at time $t = t_0$ (in other words, our strategy here is to focus on one time in the interval (a,b) at a time). Since S is a smooth surface, by Theorem 4.3.2 there exists an open set V that contains the point $\gamma(t_0)$ on the surface such that $V \cap S$ is the level set of a C^∞ function $g(x,y,z)$ whose gradient never vanishes on $V \cap S$. Since V is open, by the continuity of $\gamma(t)$ we know that there exists some small time interval $[\alpha,\beta] \subseteq (a,b)$ centered at t_0 such that $\gamma(t)$ is always inside of V, and in particular inside of $V \cap S$, for all times $t \in [\alpha,\beta]$.



Basically what we've done here is locally to $\gamma(t_0)$ we represented the surface as the level set of a C^∞ function. Why did we do this? We did this because we're about to apply the surface Euler-Lagrange vector differential equation to $\gamma(t)$ at time $t = t_0$. Let us form the arclength functional J:

$$J[u(t), v(t), w(t)] = \int_\alpha^\beta \sqrt{\left(u'(t)\right)^2 + \left(v'(t)\right)^2 + \left(w'(t)\right)^2} dt$$

where the domain of the functional is the set of C^2 curves $\left(w(t), v(t), w(t)\right)$ that lie on the surface S and that satisfy the boundary conditions of agreeing with the minimizing curve at times $t = \alpha$ and $t = \beta$:

$$\left(u(\alpha), v(\alpha), w(\alpha)\right) = \gamma(\alpha) \quad and \quad \left(u(\beta), v(\beta), w(\beta)\right) = \gamma(\beta),$$

$$\left(u'(\alpha), v'(\alpha), w'(\alpha)\right) = \gamma'(\alpha) \quad and \quad \left(u'(\beta), v'(\beta), w'(\beta)\right) = \gamma'(\beta),$$

$$\left(u''(\alpha), v''(\alpha), w''(\alpha)\right) = \gamma''(\alpha) \quad and \quad \left(u''(\beta), v''(\beta), w''(\beta)\right) = \gamma''(\beta).$$

We need all possible derivatives (0 through 2) of the curves in the domain of J to agree with the minimizing curve to make the following claim: the restriction of the minimizing curve $\gamma(t)$ to the time interval $t \in [\alpha, \beta]$ is a local minimum of the functional J. The reason why this is true should be clear (a least on an intuitive level). If it were not true that the restriction of $\gamma(t)$ to $t \in [\alpha, \beta]$ is a local minimum of J, then arbitrarily close to this restriction would exist a curve

$(\tilde{u}(t), \tilde{v}(t), \widetilde{w}(t))$ in the domain of J that has a shorter arclength than $\gamma(t)$ between the points $\gamma(\alpha)$ and $\gamma(\beta)$. But then we could construct the curve:

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & if \quad t \notin [\alpha, \beta] \\ (\tilde{u}(t), \tilde{v}(t), \widetilde{w}(t)) & if \quad t \in [\alpha, \beta] \end{cases}$$

and notice that this new curve is still C^2 (because of the above boundary conditions imposed on the curves in the domain of J, like $(\tilde{u}(t), \tilde{v}(t), \widetilde{w}(t))$) and that it has a shorter arclength than $\gamma(t)$ between the points p_1 and p_2. But this contradicts the fact that $\gamma(t)$ is a minimizing curve between p_1 and p_2. So the restriction of $\gamma(t)$ to $t \in [\alpha, \beta]$ indeed must be a local minimum of J.

Great! The fact that the restriction of $\gamma(t)$ to $t \in [\alpha, \beta]$ is a local minimum of J means that it must be a solution of the surface Euler-Lagrange vector differential equation of this arclength functional. Let $(u_0(t), v_0(t), w_0(t))$ denote the restriction of the minimizing curve $\gamma(t)$ onto the interval $t \in [\alpha, \beta]$. Let F, as in our usual notation, denote the integrand of the integral in the definition of J. Since $(u_0(t), v_0(t), w_0(t))$ is a local minimum of J, according to Theorem 3.3.1 it must satisfy the equation:[31]

$$\nabla_\delta J[u_0(t), v_0(t), w_0(t)] = \lambda(t) \nabla g(u_0(t), v_0(t), w_0(t))$$

on $t \in (\alpha, \beta)$ for some real valued function $\lambda : (\alpha, \beta) \to \mathbb{R}$. Let's calculate the left-hand side of the above equation:

$$\nabla_\delta J[u_0(t), v_0(t), w_0(t)] = \left(\frac{\partial F}{\partial u} - \frac{d}{dt}\left(\frac{\partial F}{\partial u'}\right), \frac{\partial F}{\partial v} - \frac{d}{dt}\left(\frac{\partial F}{\partial v'}\right), \frac{\partial F}{\partial w} - \frac{d}{dt}\left(\frac{\partial F}{\partial w'}\right) \right).$$

Since the integrand F does not have u, v, w in it, we have that $\frac{\partial F}{\partial u}$, $\frac{\partial F}{\partial v}$, and $\frac{\partial F}{\partial w}$ are all zero and so:

$$\nabla_\delta J[u_0(t), v_0(t), w_0(t)] = -\left(\frac{d}{dt}\left(\frac{\partial F}{\partial u'}\right), \frac{d}{dt}\left(\frac{\partial F}{\partial v'}\right), \frac{d}{dt}\left(\frac{\partial F}{\partial w'}\right) \right) = -\frac{d}{dt}\left(\frac{\partial F}{\partial u'}, \frac{\partial F}{\partial v'}, \frac{\partial F}{\partial w'}\right)$$

$$= -\frac{d}{dt}\left(\frac{u_0'}{\sqrt{{u_0'}^2 + {v_0'}^2 + {w_0'}^2}}, \frac{v_0'}{\sqrt{{u_0'}^2 + {v_0'}^2 + {w_0'}^2}}, \frac{w_0'}{\sqrt{{u_0'}^2 + {v_0'}^2 + {w_0'}^2}} \right)$$

[31] There is subtlety that is used here. Technically the way we stated Theorem 3.3.1, we need the whole surface to be the level set of a function g. But notice that that is stronger condition is never used anywhere in the proof of Theorem 3.3.1 and that its proof works equally well in the situation where every point on the locally minimizing curve is in the interior of the place on the surface where the surface is represented as the level set of such a function g. Here this is the case since every point of $(u_0(t), v_0(t), w_0(t))$, the restriction of $\gamma(t)$ to $[\alpha, \beta]$, is in the interior of V which is where the surface is represented as the level set of the function $g(x, y, z)$.

$$= -\frac{d}{dt}\left(\frac{(u_0', v_0', w_0')}{\sqrt{{u_0'}^2 + {v_0'}^2 + {w_0'}^2}}\right) = -\frac{(u_0'', v_0'', w_0'')}{\sqrt{{u_0'}^2 + {v_0'}^2 + {w_0'}^2}} - \frac{d}{dt}\left(\frac{1}{\sqrt{{u_0'}^2 + {v_0'}^2 + {w_0'}^2}}\right)(u_0', v_0', w_0').$$

Notice that $\sqrt{{u_0'}^2 + {v_0'}^2 + {w_0'}^2}$ is never zero since $\big(u_0(t), v_0(t), w_0(t)\big)$ is the restriction of the minimizing curve $\gamma(t)$ to $[\alpha, \beta]$ and by definition the derivatives of minimizing curves never vanish (see Definition 5.4.1). So we get that out surface Euler-Lagrange vector differential equation can be rewritten as:

$$-\frac{(u_0'', v_0'', w_0'')}{\sqrt{{u_0'}^2 + {v_0'}^2 + {w_0'}^2}} - \frac{d}{dt}\left(\frac{1}{\sqrt{{u_0'}^2 + {v_0'}^2 + {w_0'}^2}}\right)(u_0', v_0', w_0') = \lambda(t)\nabla g\big(u(t), v(t), w(t)\big).$$

Since $\big(u_0(t), v_0(t), w_0(t)\big)$ is the restriction of $\gamma(t)$ to $[\alpha, \beta]$, over $t \in (\alpha, \beta)$ the above equation can be rewritten as:

$$-\frac{\gamma''(t)}{\|\gamma'(t)\|} - \frac{d}{dt}(\|\gamma'(t)\|^{-1})\gamma'(t) = \lambda(t)\nabla g\big(\gamma(t)\big).$$

Solving for $\gamma''(t)$ finally gives that:

$$\gamma''(t) = -\|\gamma'(t)\|\lambda(t)\nabla g\big(\gamma(t)\big) - \|\gamma'(t)\|\frac{d}{dt}(\|\gamma'(t)\|^{-1})\gamma'(t).$$

In particular, this equation holds at time $t = t_0$:

$$\gamma''(t_0) = -\|\gamma'(t_0)\|\lambda(t_0)\nabla g\big(\gamma(t_0)\big) - \|\gamma'(t_0)\|\frac{d}{dt}(\|\gamma'(t)\|^{-1})\Big|_{t=t_0}\gamma'(t_0).$$

Now, let N be any nonzero vector that is normal to the surface at the point $\gamma(t_0)$. Let's take the vector dot product of both sides with the vector $\gamma'(t_0) \times N$:

$$\gamma''(t_0) \cdot (\gamma'(t_0) \times N)$$

$$= \left(-\|\gamma'(t_0)\|\lambda(t_0)\nabla g\big(\gamma(t_0)\big) - \|\gamma'(t_0)\|\frac{d}{dt}(\|\gamma'(t)\|^{-1})\Big|_{t=t_0}\gamma'(t_0)\right) \cdot (\gamma'(t_0) \times N).$$

By definition of the vector cross product, clearly $\gamma'(t_0) \perp (\gamma'(t_0) \times N)$. By our discussion in Section 3 of Chapter 4, we know that $\nabla g\big(\gamma(t_0)\big)$ is a perpendicular vector to the surface S at the point $\gamma(t_0)$ and so it is linearly dependent with N (they both span the space that is perpendicular to the tangent plane $T_{\gamma(t_0)}(S)$). So we have that $\nabla g\big(\gamma(t_0)\big) \perp (\gamma'(t_0) \times N)$ as well. Now, using the distributive property of the vector dot product on the right hand side of the above equation, the two facts $\gamma'(t_0) \perp (\gamma'(t_0) \times N)$ and $\nabla g\big(\gamma(t_0)\big) \perp (\gamma'(t_0) \times N)$ imply that the right hand side becomes zero:

$$\gamma''(t_0) \cdot (\gamma'(t_0) \times N) = 0.$$

Thus the minimizing curve $\gamma(t)$ satisfies the minimizing curve equation at time $t = t_0$! Since t_0 was an arbitrarily chosen time in the interval (a, b), with this we have proven that our minimizing curve $\gamma(t)$ satisfies the minimizing curve equation on all of (a, b).

■

With this we have proven the great Minimizing Curve Theorem. This theorem has a very personal connection to me because when I derived it using the calculus of variations in the summer between my second and third year at the University of Washing I finally got confirmation that the guess for the differential equation for minimizing curves that I made 1.5 years before using the principles of strings was correct. It was a triumphant moment for me!

The first time I proved the above minimizing curve theorem I did it in a very different parametrization. At the time I represented all of my surfaces almost exclusively as the graph of a function $z = f(x, y)$ and my surface curves in the form:

$$\gamma(x) = \Big(x, y(x), f\big(x, y(x)\big)\Big)$$

(this kind of representation can always be done locally to any point of a surface curve). In this parametrization the arclength functional has the form:

$$J[y] = \int_{x_0}^{x_1} \sqrt{1 + \big(y(x)\big)^2 + \left(\frac{\partial f}{\partial x}\big(x, y(x)\big) + \frac{\partial f}{\partial y}\big(x, y(x)\big) y'(x)\right)^2}\, dx.$$

I then proceeded to apply the basic Euler-Lagrange differential equation (Theorem 1.3.3) to the above functional in order to find the minimizing curve differential equation. The calculation turned out to be of frightening length and I described its nature as the following in a correspondence to someone:

> "You will get an enormous expression. In fact many of the terms in the resulting expression will cancel out. To give you an idea of how big this expression gets, approximately 262 terms cancel out. And I did it all by hand! I never used the aid of a computer to solve this problem. But I will admit one thing however that I did do some tricks in the middle of the algebra so that I didn't have to cancel out all 262 terms. In my algebra I ended up only canceling out 52 terms (I got a lot better at doing long algebraic calculations without making mistakes). But still, that was a lot of work. In the end you get that the above Euler-Lagrange Equation becomes [here the partials of f are being evaluated at $\big(x, y(x)\big)$ and I omitted writing the argument of $y(x)$]:

$$2\frac{\partial^2 f}{\partial x^2}\frac{\partial f}{\partial y} y' + 2\left(2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x^2}\frac{\partial f}{\partial x}\right) y'^2 + 2\left(\frac{\partial^2 f}{\partial y^2}\frac{\partial f}{\partial y} - 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial f}{\partial x}\right) y'^3 - 2\frac{\partial^2 f}{\partial y^2}\frac{\partial f}{\partial x} y'^4$$

$$+2\left(1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)y'y''=0.$$

A quite short equation in fact… [After some manipulation] you will get that the above equation can be turned into the form:

$$\left(-\frac{\partial f}{\partial x}\cdot\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'\right)-1\right)y''+\left(\frac{\partial f}{\partial x}y'-\frac{\partial f}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\partial y}y'+\frac{\partial^2 f}{\partial y^2}y'^2+\frac{\partial f}{\partial y}y''\right)=0."$$

I was ecstatic when I finally did this calculation at the time because it finally confirmed the guess that I made one and a half years ago about the differential equation for the minimizing curve. I later went on to think of three more proofs of the above theorem including the one I presented to you above.

Note 5.4.3: A very important equation for the minimizing curves on surfaces is in the special case when the minimizing curve is unit speed parametrized. In our definition of minimizing curve, we allowed our minimizing curve to be parametrized in any way we like as long as the derivative never vanished. However, it turns out that in many situations in differential geometry unit speed parametrizations of curves are very useful because they can often help shorten calculations that would otherwise be longer if we did them instead using non unit speed parametrizations of the same curve. This never presents an obstacle because theoretically it is not hard to show that any minimizing curve as in Definition 5.4.1 can be reparametrized to be unit speed. Indeed, let $\gamma:[a,b]\to\mathbb{R}^3$ be a minimizing curve as in Definition 5.4.1. Consider the arclength function:

$$s(t)=\int_a^t\|\gamma'(t)\|dt.$$

Since our definition of a minimizing curve requires that $\gamma'(t)$ never vanishes, we have that $\|\gamma'(t)\|$ is always strictly positive and thus the above equation implies that $s(t)$ is a strictly increasing function. Notice also that by the Fundamental Theorem of Calculus we also know that $s(t)$ is differentiable. So $s(t)$ being strictly increasing and differentiable implies that it has a strictly increasing differentiable inverse $t(s)$. Now we can form the new parametrization $\gamma\big(t(s)\big)$ of our minimizing curve and notice that now this curve is unit speed:

$$\left\|\frac{d}{ds}\Big(\gamma\big(t(s)\big)\Big)\right\|=t'(s)\big\|\gamma'\big(t(s)\big)\big\|=\frac{1}{s'\big(t(s)\big)}\big\|\gamma'\big(t(s)\big)\big\|=\frac{1}{\big\|\gamma'\big(t(s)\big)\big\|}\big\|\gamma'\big(t(s)\big)\big\|=1.$$

And notice that reparameterization of minimizing curves still leave the curve a minimizing curve since the arclength function is invariant under curve reparameterizations and the derivative of unit speed curve never vanish. So we get that all minimizing curves as in Definition 5.4.1 can be reparametrized to be unit speed.[32]

[32] In fact this trick of reparametrizing into a unit speed parametrization using the arclength function is a standard technique that works on all curves whose derivative never vanishes, not only on minimizing curves.

Great, let's then see what nice form the minimizing curve equation takes when we parametrize our minimizing curves to be unit speed. Suppose that $\gamma(t)$ is a minimizing curve with a unit speed parametrization. The minimizing curve equation that $\gamma(t)$ satisfies is:

$$\gamma''(t) \cdot \big(\gamma'(t) \times N(t)\big) = 0$$

where $N(t)$ is some vector-valued function such that at each time $t \in (a, b)$, $N(t)$ is a non-zero vector that is perpendicular to the surface at $\gamma(t)$. As I mentioned at the end of Section 2, the above equation can be reinterpreted in that it says that the vector $\gamma''(t)$ has to always lie in the plane spanned by the two vectors $\gamma'(t)$ and $N(t)$. One nice property of unit speed curves is that the second derivative $\gamma''(t)$ is always perpendicular to the tangent vector $\gamma'(t)$. The intuitive reason for this is that if $\gamma''(t)$ did have a nonzero component in the direction of $\gamma'(t)$ at some point in time, then the curve would accelerate or decelerate at that time while we said that the curve has constant speed one. Mathematically this can be shown by (the first equality below holds because $\|\gamma'(t)\|^2$ is constantly one):

$$0 = \frac{d}{dt}(\|\gamma'(t)\|^2) = \frac{d}{dt}\big(\gamma'(t) \cdot \gamma'(t)\big) = 2\gamma'(t) \cdot \gamma''(t).$$

So $\gamma'(t) \cdot \gamma''(t) = 0$ and thus they are perpendicular. So the above fact that $\gamma''(t)$ is in the span of $\gamma'(t)$ and N coupled with the fact that $\gamma''(t) \perp \gamma'(t)$ tells us that $\gamma''(t)$ is linearly dependent with N. So $\gamma''(t)$ is perpendicular to the surface! With this we get a new principle: the unit speed parametrization of a minimizing curve satisfies the property that its second derivative is constantly perpendicular to the surface.

This is an elegant formulation of minimizing curves because it gives a very simple form to the differential equation that minimizing curves must satisfy in their unit speed parametrization. It for examples allows us to write down the differential equation for minimizing curves in a surface parametrization. Let us take a unit speed minimizing curve $\gamma(t)$ and a surface parametrization Φ that the minimizing curve enters the range of for some time interval $t \in [\alpha, \beta]$. Let:

$$\big(u(t), v(t)\big) = \Phi^{-1}\big(\gamma(t)\big)$$

be the preimage of the minimizing curve under the surface parametrization for times $t \in [\alpha, \beta]$. By Lemma 4.2.10 we know that $\big(u(t), v(t)\big)$ is C^∞ and so we can do calculus on this curve. Now, let's compute $\gamma''(t)$ from the perspective of the surface parametrization. Let $N(u, v)$ denote the Gauss map $N = \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} / \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\|$. We have that (in the following, wherever I omit writing the arguments of the partials of Φ, N, and the Christoffel symbols, they are being evaluated at $\big(u(t), v(t)\big)$):

$$\gamma''(t) = \frac{d^2}{dt^2}\Big(\Phi\big(u(t), v(t)\big)\Big) = \frac{d}{dt}\left(\frac{\partial \Phi}{\partial u} u' + \frac{\partial \Phi}{\partial v} v'\right)$$

$$= \frac{\partial^2 \Phi}{\partial u^2} u'^2 + 2\frac{\partial^2 \Phi}{\partial u \partial v} u'v' + \frac{\partial^2 \Phi}{\partial v^2} v'^2 + \frac{\partial \Phi}{\partial u} u'' + \frac{\partial \Phi}{\partial v} v''$$

$$= \left(\Gamma_{11}^{1}\frac{\partial\Phi}{\partial u} + \Gamma_{11}^{2}\frac{\partial\Phi}{\partial v} - eN\right)u'^{2} + \left(\Gamma_{12}^{1}\frac{\partial\Phi}{\partial u} + \Gamma_{12}^{2}\frac{\partial\Phi}{\partial v} - fN\right)u'v'$$
$$+ \left(\Gamma_{22}^{1}\frac{\partial\Phi}{\partial u} + \Gamma_{22}^{2}\frac{\partial\Phi}{\partial v} - gN\right)v'^{2} + \frac{\partial\Phi}{\partial u}u'' + \frac{\partial\Phi}{\partial v}v''$$
$$= \left(u'' + \Gamma_{11}^{1}u'^{2} + \Gamma_{12}^{1}u'v' + \Gamma_{22}^{1}v'^{2}\right)\frac{\partial\Phi}{\partial u} + \left(v'' + \Gamma_{11}^{2}u'^{2} + \Gamma_{12}^{2}u'v' + \Gamma_{22}^{2}v'^{2}\right)\frac{\partial\Phi}{\partial v}$$
$$-\left(eu'^{2} + fu'v' + gv'^{2}\right)N.$$

The vectors $\frac{\partial\Phi}{\partial u}, \frac{\partial\Phi}{\partial v}$, and N are linearly independent and thus form a basis of $\mathbb{R}^3$ ($\frac{\partial\Phi}{\partial u}$ and $\frac{\partial\Phi}{\partial v}$ are linearly independent by condition 3 of a surface parametrization and N is linearly independent from these two since N is perpendicular to these two vectors). Now, the fact that $\gamma''(t)$ is always perpendicular to the surface, and thus linearly dependent with N, tells us that the above equation implies that the following two equations hold for all $t \in [\alpha, \beta]$:

$$u'' + \Gamma_{11}^{1}u'^{2} + \Gamma_{12}^{1}u'v' + \Gamma_{22}^{1}v'^{2} = 0,$$
$$v'' + \Gamma_{11}^{2}u'^{2} + \Gamma_{12}^{2}u'v' + \Gamma_{22}^{2}v'^{2} = 0.$$

This system of differential equations is the minimizing curve differential equation for unit speed minimizing curves. This system of differential equations is of utmost importance in differential geometry. The fact that it is expressed entirely in terms of the entries of the metric tensor (see end of Section 6 of Chapter 4) for example shows that geodesics are mapped to geodesics under isometric maps.

I remember I once saw a cartoon of Einstein writing equations on a blackboard on a National Geographic magazine and I instantly recognized the equation that he was writing. He was writing a form of the above system of differential equations for unit speed minimizing curves. He of course was writing it in physicist notation:

$$\frac{d^2\gamma^{\lambda}}{dt^2} + \Gamma_{\mu\nu}^{\lambda}\frac{d\gamma^{\mu}}{dt}\frac{d\gamma^{\nu}}{dt} = 0$$

(see: https://en.wikipedia.org/wiki/geodesic). But the meaning of this equation is exactly the same as what we wrote above. Actually, in his notation this equation applies to more general situations such as when the surface is of higher dimensions and is sitting in higher dimensional space with codimension possibly being bigger than one (terms to be defined later). But in our situation this equation in his notation distills to the exact same thing that we have. Einstein has a very specific enumeration notation which I am not particularly a fan of. For example, in the next chapter the system of differential equations for minimizing curves will take the form:

$$\forall m \in \{1,2,\dots,n\} \quad u_m'' + \sum_{k,j=1}^{n}\left(\Gamma_{k,j}^{m}u_k'u_j'\right) = 0.$$

However, in Einstein's notation one doesn't specify what the indices λ, μ, and ν run over and he doesn't even put a Σ sum symbol in his equation! To each his own I guess.

One of the cool things about learning mathematics is that as time progresses you are able to read more and more complicated mathematical texts. As a small lad I was very interested in minimizing curves and I remember visiting the Wikipedia page on minimizing curves but never understanding any of the equations. After deriving the above system of differential equations for minimizing curves, I remember that I was happy that I was finally able to start understanding those complicated looking equations that intimidated me all those years ago.

Note 5.4.4: An interesting way to verify that we're getting the right results is to see if the system of differential equations for unit speed minimizing curves in a surface parametrization indeed satisfies the Euler-Lagrange differential equations for the arclength functional expressed in terms of a surface parametrization. Let Φ be a surface parametrization. The arclength functional over the space of curves $\big(u(t), v(t)\big)$ in the domain of Φ is given by:

$$J[u(t), v(t)]$$
$$= \int_{\alpha}^{\beta} \sqrt{E\big(u(t), v(t)\big)\big(u'(t)\big)^2 + 2F\big(u(t), v(t)\big)u'(t)v'(t) + G\big(u(t), v(t)\big)\big(v'(t)\big)^2}\,dt$$

(the fact that the integral on the right-hand side is the arclength integral in a surface parametrization was discussed in Chapter 4 Section 5). Let's calculate the system of Euler-Lagrange differential equations for the above functional (Theorem 2.2.5) (in the following I omit writing the arguments of E, F, G, they are being evaluated at $\big(u(t), v(t)\big)$; I also use the Newtonian partial notation to write down the partials of E, F, and G):

$$\frac{\partial}{\partial u}\left(\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\right) - \frac{d}{dt}\left(\frac{\partial}{\partial u'}\left(\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\right)\right) = 0,$$

$$\frac{\partial}{\partial v}\left(\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\right) - \frac{d}{dt}\left(\frac{\partial}{\partial v'}\left(\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\right)\right) = 0.$$

Let's first calculate the first Euler-Lagrange differential equation above. We have that:

$$\frac{\partial}{\partial u}\left(\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\right) - \frac{d}{dt}\left(\frac{\partial}{\partial u'}\left(\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\right)\right)$$

$$= \frac{1}{2} \cdot \frac{E_u(u')^2 + 2F_u u'v' + G_u(v')^2}{\sqrt{E(u')^2 + 2Fu'v' + G(v')^2}} - \frac{d}{dt}\left(\frac{E \cdot u' + F \cdot v'}{\sqrt{E(u')^2 + 2Fu'v' + G(v')^2}}\right)$$

$$= \frac{1}{2} \cdot \frac{E_u u'^2 + 2F_u u'v' + G_u v'^2}{\sqrt{E(u')^2 + 2Fu'v' + G(v')^2}} - \frac{(E_u u' + E_v v')u' + (F_u u' + F_v v')v' + Eu'' + Fv''}{\sqrt{E(u')^2 + 2Fu'v' + G(v')^2}} +$$

$$\frac{(Eu'+Fv')\big((E_uu'+E_vv')u'^2+(F_uu'+F_vv')u'v'+(G_uu'+G_vv')v'^2\big)}{2\sqrt{E(u')^2+2Fu'v'+G(v')^2}^3}$$
$$+\frac{(Eu'+Fv')(2Eu''+Fu''v'+Fu'v''+2Gv'')}{2\sqrt{E(u')^2+2Fu'v'+G(v')^2}^3}=0.$$

Multiplying through by $2\sqrt{E(u')^2+2Fu'v'+G(v')^2}^3$ gives:

$$\begin{aligned}&\left(E_u{u'}^2+2F_uu'v'+G_u{v'}^2-2(E_uu'+E_vv')u'-2(F_uu'+F_vv')v'-2Eu''-2Fv''\right)\big(E{u'}^2\\&\quad+2Fu'v'+G{v'}^2\big)\\&\quad+(Eu'+Fv')\Big((E_uu'+E_vv'){u'}^2+(F_uu'+F_vv')u'v'+(G_uu'+G_vv'){v'}^2\\&\quad+2Eu''+Fu''v'+Fu'v''+2Gv''\Big)=0.\end{aligned}$$

Expanding all of this out and then collecting and cancelling out terms (there is a lot of cancellation that goes on: specifically 35 terms cancel out!) gives that the above equation becomes:

$$(-E_uF-E_vE+2F_uE){u'}^3v'+(2G_uE-E_uG-3E_vF+2F_uF){u'}^2{v'}^2$$
$$+(3G_uF-2E_vG-2F_vF+G_vE)u'{v'}^3+(G_uG-2F_vG+G_vF){v'}^4-2(EG-F^2)u''{v'}^2$$
$$+2(EG-F^2)u'v'v''=0.$$

Now, if we plug in the equations for the Christoffel symbols derived at the end of Section 6 of Chapter 4 you will get that:

$$\begin{aligned}&(EG-F^2)\big(2\Gamma_{11}^2{u'}^3v'+2(2\Gamma_{12}^2-\Gamma_{11}^1){u'}^2{v'}^2+2(\Gamma_{22}^2-2\Gamma_{12}^1)u'{v'}^3-2\Gamma_{22}^1{v'}^4-2u''{v'}^2\\&\quad+2u'v'v''\big)=0.\end{aligned}$$

Or dividing through by $(EG-F^2)$ gives us that:

$$2\Gamma_{11}^2{u'}^3v'+2(2\Gamma_{12}^2-\Gamma_{11}^1){u'}^2{v'}^2+2(\Gamma_{22}^2-2\Gamma_{12}^1)u'{v'}^3-2\Gamma_{22}^1{v'}^4-2u''{v'}^2+2u'v'v''=0.$$

Now if calculate the second Euler-Lagrange differential equation above you will get that:

$$2\Gamma_{22}^1u'{v'}^3+2(2\Gamma_{12}^1-\Gamma_{22}^2){u'}^2{v'}^2+2(\Gamma_{11}^1-2\Gamma_{12}^2){u'}^3v'-2\Gamma_{11}^2{u'}^4-2{u'}^2v''+2u'v'u''=0.$$

It's actually possible to arrive at the above equation without actually going through a similar calculation as above by just arguing that the symmetry of the variables u and v imply that the expanded form of the second Euler-Lagrange differential equation must have the above form. So we get that our system of Euler-Lagrange differential equations becomes:

$$2\Gamma_{11}^2{u'}^3v'+2(2\Gamma_{12}^2-\Gamma_{11}^1){u'}^2{v'}^2+2(\Gamma_{22}^2-2\Gamma_{12}^1)u'{v'}^3-2\Gamma_{22}^1{v'}^4-2u''{v'}^2+2u'v'v''=0,$$
$$2\Gamma_{22}^1u'{v'}^3+2(2\Gamma_{12}^1-\Gamma_{22}^2){u'}^2{v'}^2+2(\Gamma_{11}^1-2\Gamma_{12}^2){u'}^3v'-2\Gamma_{11}^2{u'}^4-2{u'}^2v''+2u'v'u''=0.$$

Divide through the above two equations by 2 and then a little bit of rearranging gives that the above two equations become:

$$\left(v'' + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2\right)u'v' - \left(u'' + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2\right)v'^2 = 0,$$

$$\left(u'' + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2\right)u'v' - \left(v'' + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2\right)u'^2 = 0.$$

Notice that in each equation sit the two quantities $u'' + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2$ and $v'' + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2$, which unit speed minimizing curves makes equal to zero. So all unit speed minimizing curves satisfy the Euler-Lagrange differential equation for the arclength function expressed in terms of the surface parametrization, as it should! So our theory holds.

I once wrote the above calculation in a correspondence to someone and I remember that I had trouble fitting some of the equations on some of the pages in that letter. I didn't know what to do at the time; I could have for example made the font smaller. I instead however ended up rotating the pages horizontally in the middle of the letter so that I could fit some of the equations better (including some very long fractions). As one can witness from the above calculation, differential geometry is a field that has the charm that many of its discoveries and proofs are attained by long and elegant calculations. To some this might seem like a daunting property of this field of study, but for others such calculations are a source of excitement.

Section 5: Minimal Surfaces

Another very interesting variational topic in differential geometry is the topic of minimal surfaces. Minimal surfaces, colloquially speaking, are surfaces that minimize surface area while satisfying certain boundary conditions. A simple version of the classical minimal surface problem was given in Example 2.4.12 in Chapter 2. There we were finding a necessary condition for a graph of a function that satisfied certain boundary conditions and that generated the surface of minimal surface area. The Euler-Lagrange differential equation that we derived for the surface area functional there was:

$$h_{xx}\left(1 + h_y^2\right) - 2h_x h_y h_{xy} + h_{yy}(1 + h_x^2) = 0.$$

Which remember was simplified from its equivalent from:

$$\frac{-h_{xx}\left(1 + h_y^2\right) + 2h_x h_y h_{xy} - h_{yy}(1 + h_x^2)}{\sqrt{1 + h_x^2 + h_y^2}^3} = 0.$$

By Example 4.4.5 in Chapter 4 we know that this quantity is equal to 2 times the mean curvature of the surface: $2H$. So the above equation really tells us that the minimal surface generated by the extremum h of the surface area functional has to satisfy the condition that its mean curvature is constantly equal to zero: $H \equiv 0$. This in fact almost proves the Graph Version of the Minimal Surface Theorem. In the following theorem we basically do what we did in Example 2.4.12 except here the region Ω is no longer a unit disk but a more general open set.

Theorem 5.5.1 (Graph Version of the C^2 Minimal Surface Theorem): *Let $\Omega \subseteq \mathbb{R}^2$ be a compact subset of $\mathbb{R}^2$ that Green's Theorem applies to. Suppose that Ω is the closure of its interior and that its boundary $\partial\Omega$ is a set of zero Jordan content. Suppose also that we have a function of the form $f : \partial\Omega \to \mathbb{R}$ and that the set:*

$$\{(x, y, f(x, y)) \in \mathbb{R}^3 : (x, y) \in \partial\Omega\}$$

can be parametrized by a non-singular C^2 curve $\gamma(t)$ in $\mathbb{R}^3$. This will serve as our boundary conditions for the function in the domain of our functional. Let J be the surface area functional:

$$J[h] = \iint_{\Omega} \sqrt{1 + h_x^2 + h_y^2}\,dxdy$$

whose domain is the set of functions $h \in C^2[\bar{\Omega}]$ that satisfy the boundary conditions:

$$h(x, y) = f(x, y) \quad if \quad (x, y) \in \partial\Omega$$

(in other words, the h's pass through the curve $\gamma(t)$ above $\partial\Omega$). Now, suppose that h_0 is a local minimum of the surface functional J. In other words, colloquially speaking the graph of h_0 generates a surface that minimizes surface area and that satisfies the boundary conditions of agreeing with the space curve $\gamma(t)$ at the boundary. Then, the surface generated by the graph of h_0 has constantly zero mean curvature over $\bar{\Omega}$ (here I omit writing the arguments of the partials of h_0):[33]

$$H(x, y) = -\frac{1}{2}\frac{h_{0_{xx}}\left(1 + h_{0_y}^2\right) - 2h_{0_x}h_{0_y}h_{0_{xy}} + h_{0_{yy}}\left(1 + h_{0_x}^2\right)}{\sqrt{1 + h_{0_x}^2 + h_{0_y}^2}^3} \equiv 0.$$

Proof: The proof of this theorem is exactly the same as what we did in Example 2.4.12. The Euler-Lagrange differential equation for J here is given by (Theorem 2.4.4):

$$\frac{\partial}{\partial h}\left(\sqrt{1 + h_x^2 + h_y^2}\right) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial h_x}\left(\sqrt{1 + h_x^2 + h_y^2}\right)\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial h_y}\left(\sqrt{1 + h_x^2 + h_y^2}\right)\right) = 0.$$

If you calculate out this expression as we did in Example 2.4.12 you will get that the above expression becomes:

$$\frac{-h_{xx}(1 + h_y^2) + 2h_xh_yh_{xy} - h_{yy}(1 + h_x^2)}{\sqrt{1 + h_x^2 + h_y^2}^3} = 0.$$

Multiply both sides by $1/2$ to get that:

---

[33] Technically we only defined curvatures for smooth surfaces and not surfaces generated by graphs of C^2 functions. But what we can do is just extend the definition of the surface curvatures to such C^2 surfaces by just setting their curvatures equal to the expressions for the Gaussian and mean curvatures derived at the end of Example 4.4.5. That's what we're doing here with the mean curvature.

$$-\frac{1}{2}\frac{h_{xx}\left(1+h_y^2\right)-2h_xh_yh_{xy}+h_{yy}(1+h_x^2)}{\sqrt{1+h_x^2+h_y^2}^3}=0.$$

So the local minimum h_0 must satisfy the above differential equation over Ω, of which the left-hand side is the expression for the mean curvature.

■

It is possible to prove the above theorem for smooth surfaces by considering $h \in C^\infty[\overline{\Omega}]$. We cannot however do that here automatically because technically our surface Euler-Lagrange Theorem (Theorem 2.4.4) only applies to functionals over spaces of functions in C^2. However it is not hard to generalize the proof of Theorem 2.4.4 so that it applies to $h \in C^\infty$ and in fact the only thing that's needed to do this is to prove a C^∞ version of Lemma 2.4.2. The way this is done is by replacing the m in that lemma with ∞ and thinking of an equation for a C^∞ spike in the proof of that lemma. There are many ways to write down an equation for a C^∞ spike centered at a point $(x_0, y_0) \in \mathbb{R}^2$ and one classical way to do such a thing is by setting:

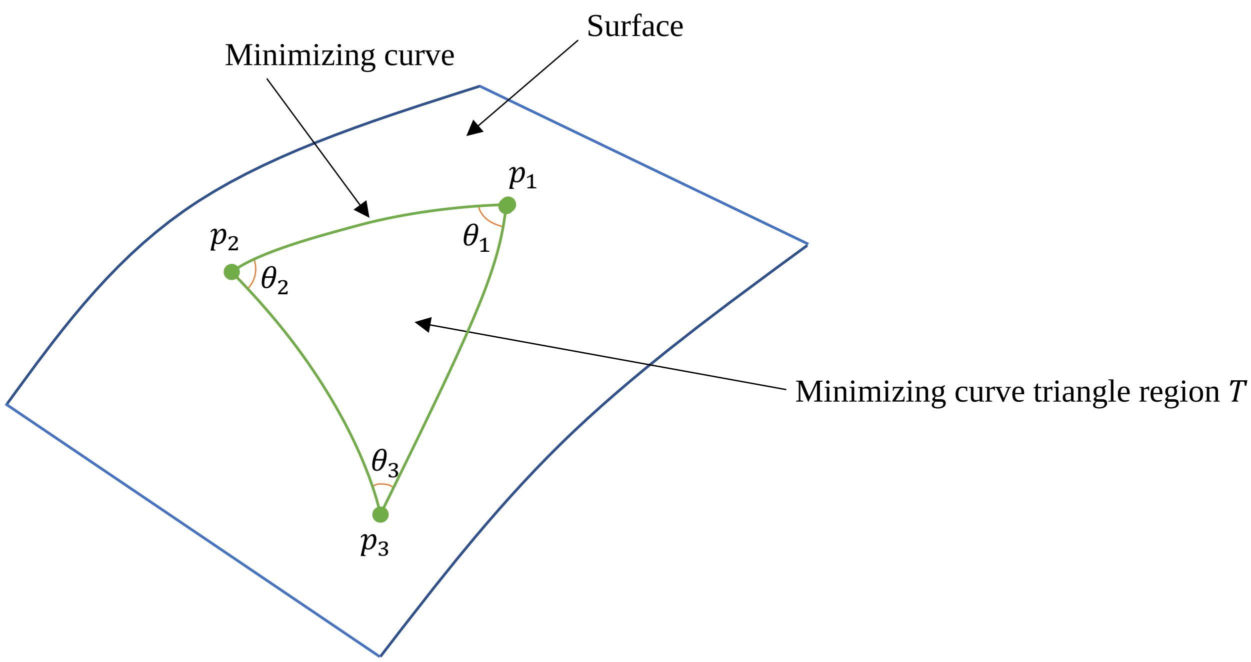
$$h(x,y)=\begin{cases} e^{\frac{1}{(x-x_0)^2+(y-y_0)^2-r^2}} & if \quad (x,y)\in B_r(x_0,y_0) \\ 0 & if \quad (x,y)\notin B_r(x_0,y_0) \end{cases}.$$

And a similar thing goes for the higher dimensional version of Theorem 2.4.4 (Theorem 2.4.8).

Minimal surfaces appear in the study of soap films because soap films in the absence of pressure configure themselves to take the form of minimal surfaces. The physical reason for this is that the surface energy of a soap film is proportional to its total surface area and so soap films in their effort to minimize their total surface energy try to minimize their total surface area. Demonstration of this phenomenon can be done by taking a closed looped metal wire, deforming it, submerging it into a soap fluid, and then taking it out. We all did a special case of this by taking a small circular hoop (usually on the end of a stick), submerging it into bubble fluid, and then taking it out to blow on it in order to form bubbles. Remember that as you take out the circular hoop you get that the fluid forms the surface of a portion of a plane. Indeed this is so because the minimal surface that satisfies the boundary condition of intersecting some set that lies in some plane (such as a circular hoop) is a portion of a plane itself. Another such interesting demonstration is if you use the soap fluid to form a soap film that goes between two circular hoops. If the circular hoops are positioned just right, you will get a catenoid surface of revolution. These surfaces also turn out to have mean curvature zero.

Section 6: Global Gauss-Bonnet Theorem

The Gaussian curvature turns out to have many magical abilities when it comes to describing the nature of surfaces. Yes, describing the curvature a surface is one ability in its repertoire, but it turns out that it can do so much more. In my opinion the most beautiful theorem that I've seen involving the Gaussian curvature is that the difference of the sum of the angles (in radians) of a small surface minimizing curve triangle and π is equal to the surface integral of the Gaussian curvature within the triangle. This is called "Gauss's Theorem on the Sum of the Internal Angles of a Small Minimizing Curve Triangle." A minimizing curve triangle is defined as taking three points on a surface and connecting each pair of points by their corresponding minimizing curves. The resulting region will be a surface triangle, something like:



Gauss's theorem then says that:

$$\theta_1 + \theta_2 + \theta_3 - \pi = \iint_T K \, d\sigma.$$

where the integrand of the surface integral on the right-hand side is the Gaussian curvature as a function of points on the surface (in other words, the right-hand side is the surface integral of the Gaussian curvature over T). This a powerful theorem that is not easy to prove and a slight generalization of this is a fundamental result that's used to prove this theorem on more global scales and on more complicated regions such as "surface polygons."

Minimizing curve triangles are very natural generalizations of the notion of triangles. In normal Euclidian 2-dimensional geometry, what is a triangle? A triangle is a figure with three sides where the three sides are lines. And what are lines in the plane? They are paths of minimal arclength. In other words, the three sides of a triangle in normal Euclidean geometry are minimizing curves. So it is a natural choice to define triangles on surfaces as triangles whose sides are minimizing curves. It turns out that in differential geometry other types of surface triangles are very interesting to consider and so in order to be precise, we call such triangles "minimizing curve triangles."

Gaussian curvature surprisingly also has the ability to describe the global characteristic of a closed surface. Intuitively this ability is described as follows. The **total Gaussian curvature** of a surface S is defined as the surface integral of the Gaussian curvature over the entire surface:

$$K[S] = \oiint_S K \, d\sigma.$$

Take any surface S that has both an interior and an exterior, such as a sphere, ellipsoid, or a torus. Then the total Gaussian curvature $K[S]$ can only be a value in the following discrete set:

$$4\pi, 0, -4\pi, -8\pi, \ldots$$

And intuitively the rule for what the value of the total curvature of a closed surface is given by the following:

1.) If the surface S can be obtained by continuously deforming and stretching the sphere, then its total Gaussian curvature $K[S] = 4\pi$.
2.) If the surface S can be obtained by continuously deforming and stretching the torus, then its total Gaussian curvature $K[S] = 0$.
3.) If the surface S can be obtained by continuously deforming and stretching the double torus (torus with two holes), then its total Gaussian curvature $K[S] = -4\pi$.
4.) If the surface S can be obtained by continuously deforming and stretching the triple torus (torus with three holes), then its total Gaussian curvature $K[S] = -8\pi$.

$\vdots$

(the rule goes on following the pattern: quadruple torus, quintuple torus, etc.). The precise statement and proof of the above fact is beyond difficult and is a corollary of a theorem called the Global-Gauss Bonnet Theorem. The above is really an amazing fact because it partly says that no matter how your closed surface looks like, the Gaussian curvature will distribute itself along the surface so that the integral of the Gaussian curvature over the whole surface will always be a member of the discrete set $\{4\pi, 0, -4\pi, -8\pi, \ldots\}$.

We will prove a bit weaker form of the above fact that will yield to our variational theory. Intuitively what we're going to prove is that for any closed surface that can be obtained from one of the above fundamental surfaces (i.e. the sphere, the torus, the double torus, etc.) via many local deformations, the total Gaussian curvature of the surface will be the same as that original fundamental surface. And they way that we're going to prove this is by showing that the total Gaussian curvature is invariant under such local deformations.

The local deformations that we will be working with will involve locally deforming surfaces using graphs. We shall call such local deformations "smooth local graph deformations" and intuitively all they do is they deform the surface in a small neighborhood using flows written in forms of graphs. Let's precisely define them as follows.

Definition 5.6.1: *Let S be a smooth 2-dimensional surface sitting in $\mathbb{R}^3$ and $p \in S$ be any point on it. In an open neighborhood V of p, S can be represented as the graph of a C^∞ function of the form $f : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}^2$ is some open set. Let's suppose that f is of the form $z = f(x, y)$ (this definition is similar in the other two cases $y = f(x, z)$ and $x = f(y, z)$). Let's also suppose that our open set V is of the form $U \times (\alpha, \beta)$ (a set theoretic cylinder), which can always be arranged by choosing our U and (α, β) small enough.*

Now, let p_{xy} be the projection of p onto the x-y plane. Since $p_{xy} \in U$ and U is open, there exists an open ball $B_r(p_{xy})$ centered at p_{xy} such that $B_r(p_{xy}) \subseteq U$. Ok, let $g \in C^\infty[U]$ be any function such that g vanishes outside of $B_{r/2}(p_{xy})$ (note the $r/2$). Let $\Lambda : U \times [0, a] \to \mathbb{R}$ be the ∞-smooth linear flow defined by:

$$\Lambda(x, y, t) = f(x, y) + g(x, y)t.$$

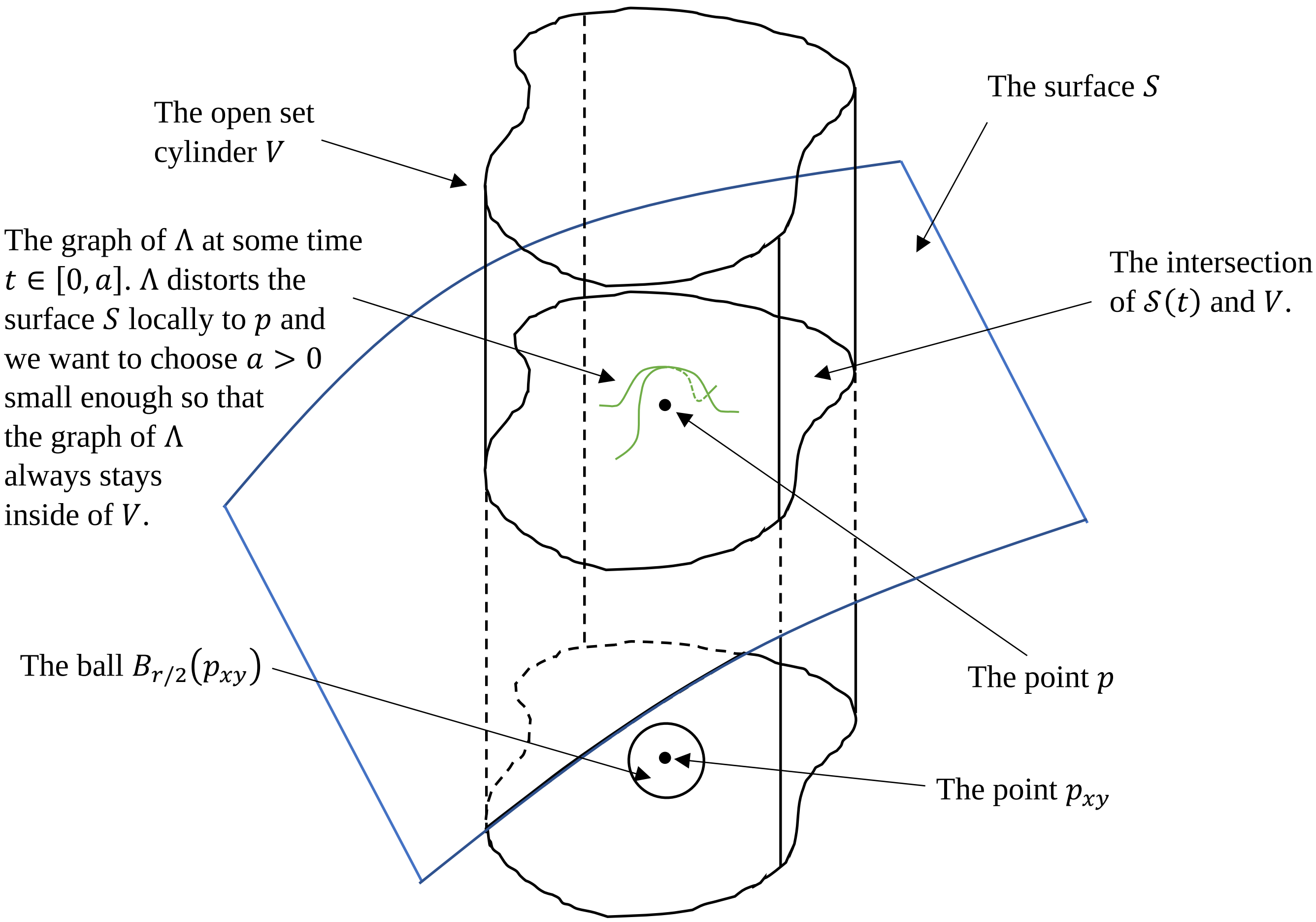
where for each $t \in [0, a]$ the graph of $\Lambda(x, y)$ is inside of V (we will prove after this definition that we can always choose $a > 0$ small enough so that this holds). Finally, for each time $t \in [0, a]$ let $\mathcal{S}(t)$ be the surface defined by:

$$\mathcal{S}(t) = (S \setminus V) \cup \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in U \quad and \quad z = \Lambda(x, y, t)\}.$$

The function $\mathcal{S}(t)$ is called a ***smooth local graph deformation*** *of S. The surface $\mathcal{S}(a)$ is the surface that $S = \mathcal{S}(0)$ deforms to under this deformation.*[34]

Wow, that was a lot of technical confusion! Let's break it down a little bit. In the above definition we started off with a smooth surface S and a point $p \in S$ on it. We then took an open set theoretic cylinder (kind of like a cylinder except that the base is some open set rather than a disk) as an open neighborhood of p in which the surface is the graph of a function $f : U \to \mathbb{R}$ of the form $z = f(x, y)$ (the above definition in the cases $y = f(x, z)$ and $x = f(y, z)$ are defined similarly).

[34] In the other two cases of where f is of the form $y = f(x, z)$ or $x = f(y, z)$, the formula for $\mathcal{S}(t)$ is similar. They are respectively $\mathcal{S}(t) = (S \setminus V) \cup \{(x, y, z) \in \mathbb{R}^3 : (x, z) \in U \quad and \quad y = \Lambda(x, z, t)\}$ and $\mathcal{S}(t) = (S \setminus V) \cup \{(x, y, z) \in \mathbb{R}^3 : (y, z) \in U \quad and \quad x = \Lambda(y, z, t)\}$ where U are in their appropriate spaces.



Then we took any C^∞ function g over U that satisfied the boundary conditions of vanishing outside the small ball $B_{r/2}(p_{xy})$ and then we considered the linear flow $\Lambda(x, y, t) = f(x, y) + g(x, y)t$ for t in a small enough interval $[0, a]$. Small enough meant that we wanted the graph of the flow to always remain inside of the above cylinder. The reason why we can choose such a small enough $a > 0$ to satisfy this is the following. The closest distance that the graph of f over $B_{r/2}(p_{xy})$ ever gets to the "floor" or the "ceiling" of the set theoretic cylinder V is given by (remember the z-values of the floor and the ceiling of V are α and β [see again the above definition]):

$$d = \min\left\{\underline{\min}_{(x,y)\in B_{r/2}(p_{xy})}\{|f(x, y) - \alpha|\}, \underline{\min}_{(x,y)\in B_{r/2}(p_{xy})}\{|\beta - f(x, y)|\}\right\}.$$

By the Extreme Value Theorem, we know that this quantity exists and that there exists a point $(x_0, y_0) \in \overline{B_{r/2}(p_{xy})}$ such that either:

$$|f(x_0, y_0) - \alpha| = d \qquad or \qquad |\beta - f(x_0, y_0)| = d.$$

Since $(x_0, y_0, f(x_0, y_0))$ lies in the interior of V it's contained in some ball contained in V and thus $d > 0$. In other words, we have that both the "distance" between the graph of f and the floor of V and the "distance" between the graph of f and the ceiling of V are bigger than some positive constant (d to be specific). Now, as you start to let t vary from $t = 0$, the closest the graph of $\Lambda(x, y, t)$ will ever get to the floor or the ceiling of V is similarly given by:

$$\min\left\{\underline{\min_{(x,y)\in B_{r/2}(p_{xy})}}\{|\Lambda(x,y,t)-\alpha|\},\ \underline{\min_{(x,y)\in B_{r/2}(p_{xy})}}\{|\beta-\Lambda(x,y,t)|\}\right\} =$$

$$\min\left\{\underline{\min_{(x,y)\in B_{r/2}(p_{xy})}}\{|f(x,y)+g(x,y)t-\alpha|\},\ \underline{\min_{(x,y)\in B_{r/2}(p_{xy})}}\{|\beta-(f(x,y)+g(x,y)t)|\}\right\}.$$

At $t = 0$ this quantity is equal to $d > 0$. And since this quantity varies continuously in terms of t, we get that we can choose a small enough $a > 0$ such that for $t \in [0, a]$ the above quantity still remains positive. This means that for all $t \in [0, a]$, the graph of the flow $\Lambda(x, y, t)$ will always stay inside of the set theoretic cylinder V and thus such an $a > 0$ indeed always exists.

From here in the above definition we formed the function $\mathcal{S}(t)$ which for every time $t \in [0, a]$, $\mathcal{S}(t)$ was the set:

$$\mathcal{S}(t) = (S \setminus V) \cup \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in U \quad and \quad z = \Lambda(x, y, t)\}.$$

This is really just the set obtained by taking the surface S outside of V and taking its union with the graph of $\Lambda(x, y, t)$ over U. By using the local graph definition of a smooth surface, it shouldn't be too hard to see that for each time $t \in [0, a]$, $\mathcal{S}(t)$ is a smooth surface. This really comes from the fact that for each such time t, $\Lambda(x, y, t)$ matches up with f outside of $B_{r/2}(p_{xy})$ (since g vanishes outside of this ball) and $B_r(p_{xy}) \subseteq U$ and thus the graph of $\Lambda(x, y, t)$ connects smoothly with the portion of the surface S outside of V: $S \setminus V$.[35] I will leave the filling of the details of verifying that $\mathcal{S}(t)$ is a smooth surface to the reader (it shouldn't be too hard, look at the local graph representation of $\mathcal{S}(t)$ at each point on it; Draw a picture like above!).

Notice also the important observation that $\mathcal{S}(0) = S$. Visually if you would play an animation of what $\mathcal{S}(t)$ looks like as t runs from $t = 0$ to $t = a$ you would essentially see our original surface S being locally deformed at p and that $\mathcal{S}(t)$ is S at $t = 0$ and it smoothly deforms to the surface $\mathcal{S}(a)$ at $t = a$.

With this idea of a deformation, we are now ready to prove a version of the above corollary of the Global Gauss-Bonnet Theorem.

Theorem 5.6.2 (Invariance of the Total Gaussian Curvature under Smooth Local Graph Deformations): *Suppose that S is a smooth 2-dimensional surface sitting in $\mathbb{R}^3$ and that $\mathcal{S}(t)$ is a smooth local graph deformation of S defined over the interval $t \in [0, a]$ where $a > 0$. Then, for any $t \in [0, a]$, the total Gaussian curvature of $\mathcal{S}(t)$ is equal to:*

$$K[\mathcal{S}(t)] = K[S].$$

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[35] This is in fact why I concentrated on the bit smaller ball $B_{r/2}(p_{xy})$ rather than the ball $B_r(p_{xy})$ which merely fits inside of U. This way $\Lambda(x, y, t)$ has "room" in $B_r(p_{xy}) \setminus B_{r/2}(p_{xy})$ to smoothly connect with the surface S outside of V and thus $\mathcal{S}(t)$ is a smooth surface, which can be seen by looking at this surface locally at each point using graphs of smooth functions.

In other words, deforming a surface using smooth local graph deformations doesn't change its total Gaussian curvature.

Proof: Let's suppose that our local graph deformation $\mathcal{S}(t)$ is of the form:

$$\mathcal{S}(t) = (S \setminus V) \cup \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in U \quad and \quad z = \Lambda(x, y, t)\}$$

(this proof in the other two cases mentioned in Footnote 32 are done similarly). Let's carry over all of the notation that we had Definition 5.6.1 into this proof. In other words, let $B_r(p_{xy}) \subseteq U$, $f, g \in C^\infty[U]$ such that g vanishes outside of $B_{r/2}(p_{xy})$, and:

$$\Lambda(x, y, t) = f(x, y) + g(x, y)t.$$

At $t = 0$, it's obvious that $K[\mathcal{S}(0)] = K[S]$ since $\mathcal{S}(0) = S$. So let's prove that $K[\mathcal{S}(t)]$ is constantly equal to $K[S]$ by just showing that its time derivative is constantly zero.[36] We have that:

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \frac{d}{dt}\left(\oiint_{\mathcal{S}(t)} K \, d\sigma\right).$$

It's interesting to note that the task of calculating this derivative and showing that it is equal to zero is fundamentally equivalent to showing that the Gaussian curvature is unchanging under such a deformation. Outside of V our surface is not deforming in any way and so the total Gaussian curvature there is unchanging. In other words, since there is no deformation happening outside of V we have that:

$$\frac{d}{dt}\left(\oiint_{\mathcal{S}(t)\setminus V} K \, d\sigma\right) = 0.$$

So really, we can just rewrite the expression for the time derivative of $K[\mathcal{S}(t)]$ as:

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \frac{d}{dt}\left(\oiint_{\mathcal{S}(t)\setminus V} K \, d\sigma + \oiint_{\mathcal{S}(t)\cap V} K \, d\sigma\right) = \frac{d}{dt}\left(\oiint_{\mathcal{S}(t)\cap V} K \, d\sigma\right).$$

This is nice because now we expressed the time derivative of $K[\mathcal{S}(t)]$ as an integral over a local piece of the surface which by the way can be represented as the graph of the function $\Lambda(x, y, t)$ for every fixed t. In explanation, the piece of the surface $\mathcal{S}(t) \cap V$ is the graph of $\Lambda(x, y, t)$ over U and so we can rewrite the last surface integral in the above expression as (here I use the Newtonian notation for partial derivatives):

[36] For some of you, the theme of the methods of showing that two functionals are equal will be ringing here. I'll mention something about this after the proof.

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \frac{d}{dt}\left(\iint\limits_U K\sqrt{1+\left(\Lambda_x(x,y,t)\right)^2+\left(\Lambda_y(x,y,t)\right)^2}\,dxdy\right).$$

Plugging in the formula for the Gaussian curvature of a graph that we derived at the end of Section 4 of Chapter 4, we get that the above integral can be rewritten as (here I omit the argument of Λ and its partials; they are being evaluated at (x, y, t)):

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \frac{d}{dt}\left(\iint\limits_U \frac{\Lambda_{xx}\Lambda_{yy}-\Lambda_{xy}^2}{\left(1+\Lambda_x^2+\Lambda_y^2\right)^2}\sqrt{1+\Lambda_x^2+\Lambda_y^2}\,dxdy\right)$$

$$= \frac{d}{dt}\left(\iint\limits_U \frac{\Lambda_{xx}\Lambda_{yy}-\Lambda_{xy}^2}{\sqrt{1+\Lambda_x^2+\Lambda_y^2}^3}\,dxdy\right).$$

Since we're going to carry the derivative under the integral sign we, it would be nice to have our domain of integration compact. Notice that the surface does not change outside of the ball $B_{3r/4}(p_{xy})$ since g vanishes outside of $B_{r/2}(p_{xy})$ ($3r/4$ is just some number I chose between $r/2$ and r; we'll soon see why we need such a radius). So the above expression is in fact equivalent to:

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \frac{d}{dt}\left(\iint\limits_{\overline{B_{3r/4}(p_{xy})}} \frac{\Lambda_{xx}\Lambda_{yy}-\Lambda_{xy}^2}{\sqrt{1+\Lambda_x^2+\Lambda_y^2}^3}\,dxdy\right).$$

Now that we have a compact region of integration, we will be able to carry the derivative under the integral sign. If you just carry the derivative under the integral sign right now and compute the derivative of the integrand, it will not be immediately clear that the above quantity is zero. Indeed, you will not get that the derivative of the integrand is zero everywhere since the Gaussian curvature is changing at each point under the surface deformation. In order to show that the above quantity is zero you have to use the above integral's behavior over the whole region of integration and the crucial fact that the flow Λ satisfies the smooth boundary conditions of being completely unchanging near the boundary of $\overline{B_{3r/4}(p_{xy})}$. We have in fact already done this sort of analysis in Chapter 3 in Section 7 when we used the concept of the functional derivative in order to compute quantities like above. So let's apply those results here. A nice way to describe the above derivative is to instead consider the following local "total Gaussian curvature functional:"

$$J[h] = \iint\limits_{\overline{B_{3r/4}(p_{xy})}} \frac{h_{xx}h_{yy}-h_{xy}^2}{\sqrt{1+h_x^2+h_y^2}^3}\,dxdy$$

defined over the space of functions $h \in C^\infty\left[\overline{B_{r/2}(p_{xy})}\right]$ that satisfy the boundary conditions of being equal to $f(x, y)$ on the boundary $\partial B_{r/2}(p_{xy})$ and that all their first order partials are equal to that of $f(x, y)$ on this boundary. Notice that we can now reformulate the equation above for the time derivative of $K[\mathcal{S}(t)]$ as:

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \frac{d}{dt}(J[\Lambda]).$$

It's easy to check that the restriction of Λ to $\overline{B_{r/2}(p_{xy})}$ is a flow that stays inside of the domain J (to see this just relook at the equation for Λ in terms of f and g again). But look, we know an expression for the quantity on the right because we derived it in Theorem 3.7.1. By Theorem 3.7.1 we have that:[37]

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \iint\limits_{\overline{B_{3r/4}(p_{xy})}} \frac{\delta J}{\delta h}[\Lambda]\Lambda_t dxdy.$$

So if we show that $\frac{\delta J}{\delta h}[\Lambda] \equiv 0$ on $t \in [0, a]$, then we will get that the above equation implies that the time derivative of $K[\mathcal{S}(t)]$ is also constantly equal to zero. So let's show that $\frac{\delta J}{\delta h} \equiv 0$. This task of showing the variational derivative of the local "total Gaussian curvature functional" is equal to zero is in fact the variational theme behind the statement that the total Gaussian curvature is invariant under local deformations. Let's calculate the variational derivative of J evaluated at any general h sitting inside of this functional's domain. We have that:

$$\frac{\delta J}{\delta h}[h] = \frac{\partial F}{\partial h} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial h_{xx}}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial h_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial h_{yy}}\right)$$

where F is the integrand of the integral in the expression that defines J above. Let's break down the computation of each term on the right-hand side separately. In what follows I will use the equality of mixed partials several times.

Calculating the $\frac{\partial F}{\partial h}$ term gives:

$$\frac{\partial F}{\partial h} = 0.$$

Calculating the $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right)$ term gives:

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial h_x}\right) = \frac{\partial}{\partial x}\left(-3\frac{(h_{xx}h_{yy} - h_{xy}^2)h_x}{\sqrt{1 + h_x^2 + h_y^2}^5}\right)$$

$$= -3\frac{(h_{xxx}h_{yy} + h_{xx}h_{yyx} - 2h_{xy}h_{xyx})h_x + (h_{xx}h_{yy} - h_{xy}^2)h_{xx}}{\sqrt{1 + h_x^2 + h_y^2}^5}$$

[37] I'll remind you that the proof of Theorem 3.7.1 involves carrying a derivative under the integral sign. Notice that here we are satisfying the region of integration being compact assumption in that lemma.

$$+15\frac{\left(h_{xx}h_{yy}-h_{xy}^2\right)h_x\left(h_xh_{xx}+h_yh_{yx}\right)}{\sqrt{1+h_x^2+h_y^2}^7}.$$

Calculating the $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right)$ term gives:

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial h_y}\right)=\frac{\partial}{\partial y}\left(-3\frac{\left(h_{xx}h_{yy}-h_{xy}^2\right)h_y}{\sqrt{1+h_x^2+h_y^2}^5}\right)$$

$$=-3\frac{\left(h_{xxy}h_{yy}+h_{xx}h_{yyy}-2h_{xy}h_{xyy}\right)h_y+\left(h_{xx}h_{yy}-h_{xy}^2\right)h_{yy}}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$+15\frac{\left(h_{xx}h_{yy}-h_{xy}^2\right)h_y\left(h_xh_{xy}+h_yh_{yy}\right)}{\sqrt{1+h_x^2+h_y^2}^7}.$$

Calculating the $\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial h_{xx}}\right)$ term gives:

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial h_{xx}}\right)=\frac{\partial^2}{\partial x^2}\left(\frac{h_{yy}}{\sqrt{1+h_x^2+h_y^2}^3}\right)=\frac{\partial}{\partial x}\left(\frac{h_{yyx}}{\sqrt{1+h_x^2+h_y^2}^3}-3\frac{h_{yy}\left(h_xh_{xx}+h_yh_{yx}\right)}{\sqrt{1+h_x^2+h_y^2}^5}\right)$$

$$=\frac{h_{yyxx}}{\sqrt{1+h_x^2+h_y^2}^3}-3\frac{h_{yyx}\left(h_xh_{xx}+h_yh_{yx}\right)}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$-3\frac{h_{yyx}\left(h_xh_{xx}+h_yh_{yx}\right)+h_{yy}\left(h_{xx}^2+h_xh_{xxx}+h_{xy}^2+h_yh_{yxx}\right)}{\sqrt{1+h_x^2+h_y^2}^5}+15\frac{h_{yy}\left(h_xh_{xx}+h_yh_{yx}\right)^2}{\sqrt{1+h_x^2+h_y^2}^7}.$$

Calculating the $\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial h_{xy}}\right)$ term gives:

$$\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial h_{xy}}\right)=\frac{\partial^2}{\partial x\partial y}\left(-2\frac{h_{xy}}{\sqrt{1+h_x^2+h_y^2}^3}\right)$$

$$=-2\frac{\partial}{\partial y}\left(\frac{h_{xyy}}{\sqrt{1+h_x^2+h_y^2}^3}-3\frac{h_{xy}\left(h_xh_{xy}+h_yh_{yy}\right)}{\sqrt{1+h_x^2+h_y^2}^5}\right)$$

$$=-2\frac{h_{xyyx}}{\sqrt{1+h_x^2+h_y^2}^3}+6\frac{h_{xyy}\left(h_xh_{xx}+h_yh_{yx}\right)}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$+6\frac{h_{xyx}\left(h_x h_{xy}+h_y h_{yy}\right)+h_{xy}\left(h_{xx}h_{xy}+h_x h_{xyx}+h_{yx}h_{yy}+h_y h_{yyx}\right)}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$-30\frac{h_{xy}\left(h_x h_{xy}+h_y h_{yy}\right)\left(h_x h_{xx}+h_y h_{yx}\right)}{\sqrt{1+h_x^2+h_y^2}^7}.$$

Calculating the $\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial h_{yy}}\right)$ term gives:

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial h_{yy}}\right)=\frac{\partial^2}{\partial y^2}\left(\frac{h_{xx}}{\sqrt{1+h_x^2+h_y^2}^3}\right)=\frac{\partial}{\partial y}\left(\frac{h_{xxy}}{\sqrt{1+h_x^2+h_y^2}^3}-3\frac{h_{xx}\left(h_x h_{xy}+h_y h_{yy}\right)}{\sqrt{1+h_x^2+h_y^2}^5}\right)$$

$$=\frac{h_{xxyy}}{\sqrt{1+h_x^2+h_y^2}^3}-3\frac{h_{xxy}\left(h_x h_{xy}+h_y h_{yy}\right)}{\sqrt{1+h_x^2+h_y^2}}$$

$$-3\frac{h_{xxy}\left(h_x h_{xy}+h_y h_{yy}\right)+h_{xx}\left(h_{xy}^2+h_x h_{xyy}+h_{yy}^2+h_y h_{yyy}\right)}{\sqrt{1+h_x^2+h_y^2}^5}+15\frac{h_{xx}\left(h_x h_{xy}+h_y h_{yy}\right)^2}{\sqrt{1+h_x^2+h_y^2}^7}.$$

Plugging all of these into the expression for $\frac{\delta J}{\delta h}[h]$ gives us that:

$$\frac{\delta J}{\delta h}[h]=3\frac{\left(h_{xxx}h_{yy}+h_{xx}h_{yyx}-2h_{xy}h_{xyx}\right)h_x+\left(h_{xx}h_{yy}-h_{xy}^2\right)h_{xx}}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$-15\frac{\left(h_{xx}h_{yy}-h_{xy}^2\right)h_x\left(h_x h_{xx}+h_y h_{yx}\right)}{\sqrt{1+h_x^2+h_y^2}^7}$$

$$+3\frac{\left(h_{xxy}h_{yy}+h_{xx}h_{yyy}-2h_{xy}h_{xyy}\right)h_y+\left(h_{xx}h_{yy}-h_{xy}^2\right)h_{yy}}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$-15\frac{\left(h_{xx}h_{yy}-h_{xy}^2\right)h_y\left(h_x h_{xy}+h_y h_{yy}\right)}{\sqrt{1+h_x^2+h_y^2}^7}+\frac{h_{yyxx}}{\sqrt{1+h_x^2+h_y^2}^3}-3\frac{h_{yyx}\left(h_x h_{xx}+h_y h_{yx}\right)}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$-3\frac{h_{yyx}\left(h_x h_{xx}+h_y h_{yx}\right)+h_{yy}\left(h_{xx}^2+h_x h_{xxx}+h_{xy}^2+h_y h_{yxx}\right)}{\sqrt{1+h_x^2+h_y^2}^5}+15\frac{h_{yy}\left(h_x h_{xx}+h_y h_{yx}\right)^2}{\sqrt{1+h_x^2+h_y^2}^7}$$

$$-2\frac{h_{xyyx}}{\sqrt{1+h_x^2+h_y^2}^3}+6\frac{h_{xyy}\left(h_x h_{xx}+h_y h_{yx}\right)}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$+6\frac{h_{xyx}\left(h_x h_{xy}+h_y h_{yy}\right)+h_{xy}\left(h_{xx}h_{xy}+h_x h_{xyx}+h_{yx}h_{yy}+h_y h_{yyx}\right)}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$-30\frac{h_{xy}(h_xh_{xy}+h_yh_{yy})(h_xh_{xx}+h_yh_{yx})}{\sqrt{1+h_x^2+h_y^2}^7}+\frac{h_{xxyy}}{\sqrt{1+h_x^2+h_y^2}^3}-3\frac{h_{xxy}(h_xh_{xy}+h_yh_{yy})}{\sqrt{1+h_x^2+h_y^2}}$$

$$-3\frac{h_{xxy}(h_xh_{xy}+h_yh_{yy})+h_{xx}(h_{xy}^2+h_xh_{xyy}+h_{yy}^2+h_yh_{yyy})}{\sqrt{1+h_x^2+h_y^2}^5}+15\frac{h_{xx}(h_xh_{xy}+h_yh_{yy})^2}{\sqrt{1+h_x^2+h_y^2}^7}.$$

A quite formidable looking expression. We have to show that it is equal to zero! Nicely a lot of the terms immediately cancel. For those of you who are reading the color version of this book I highlighted in different colors groups of terms that cancel each other out in the above expression. Canceling those terms out gives:

$$\frac{\delta J}{\delta h}[h]=3\frac{(h_{xx}h_{yyx}-2h_{xy}h_{xyx})h_x+(h_{xx}h_{yy}-h_{xy}^2)h_{xx}}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$-15\frac{(h_{xx}h_{yy}-h_{xy}^2)h_x(h_xh_{xx}+h_yh_{yx})}{\sqrt{1+h_x^2+h_y^2}^7}+3\frac{(h_{xxy}h_{yy}-2h_{xy}h_{xyy})h_y+(h_{xx}h_{yy}-h_{xy}^2)h_{yy}}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$-15\frac{(h_{xx}h_{yy}-h_{xy}^2)h_y(h_xh_{xy}+h_yh_{yy})}{\sqrt{1+h_x^2+h_y^2}^7}-3\frac{h_{yy}(h_{xx}^2+h_{xy}^2+h_yh_{yxx})}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$+15\frac{h_{yy}(h_xh_{xx}+h_yh_{yx})^2}{\sqrt{1+h_x^2+h_y^2}^7}+6\frac{h_{xy}(h_{xx}h_{xy}+h_xh_{xyx}+h_{yx}h_{yy}+h_yh_{yyx})}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$-30\frac{h_{xy}(h_xh_{xy}+h_yh_{yy})(h_xh_{xx}+h_yh_{yx})}{\sqrt{1+h_x^2+h_y^2}^7}-3\frac{h_{xx}(h_{xy}^2+h_xh_{xyy}+h_{yy}^2)}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$+15\frac{h_{xx}(h_xh_{xy}+h_yh_{yy})^2}{\sqrt{1+h_x^2+h_y^2}^7}.$$

I colored in more terms above that cancel each other out. Canceling these terms out gives:

$$\frac{\delta J}{\delta h}[h]=-3\frac{h_{xy}^2h_{xx}}{\sqrt{1+h_x^2+h_y^2}^5}-15\frac{(h_{xx}h_{yy}-h_{xy}^2)h_x(h_xh_{xx}+h_yh_{yx})}{\sqrt{1+h_x^2+h_y^2}^7}$$

$$-3\frac{2h_{xy}h_{xyy}h_y+h_{xy}^2h_{yy}}{\sqrt{1+h_x^2+h_y^2}^5}-15\frac{(h_{xx}h_{yy}-h_{xy}^2)h_y(h_xh_{xy}+h_yh_{yy})}{\sqrt{1+h_x^2+h_y^2}^7}-3\frac{h_{yy}h_{xy}^2}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$+15\frac{h_{yy}(h_xh_{xx}+h_yh_{yx})^2}{\sqrt{1+h_x^2+h_y^2}^7}+6\frac{h_{xy}(h_{xx}h_{xy}+h_{yx}h_{yy}+h_yh_{yyx})}{\sqrt{1+h_x^2+h_y^2}^5}$$

$$-30\frac{h_{xy}\left(h_xh_{xy}+h_yh_{yy}\right)\left(h_xh_{xx}+h_yh_{yx}\right)}{\sqrt{1+h_x^2+h_y^2}^7}-3\frac{h_{xx}h_{xy}^2}{\sqrt{1+h_x^2+h_y^2}^5}+15\frac{h_{xx}\left(h_xh_{xy}+h_yh_{yy}\right)^2}{\sqrt{1+h_x^2+h_y^2}^7}.$$

I colored in some more terms above that cancel each other out. Canceling these terms out gives:

$$\frac{\delta J}{\delta h}[h]=-15\frac{\left(h_{xx}h_{yy}-h_{xy}^2\right)h_x\left(h_xh_{xx}+h_yh_{yx}\right)}{\sqrt{1+h_x^2+h_y^2}^7}-15\frac{\left(h_{xx}h_{yy}-h_{xy}^2\right)h_y\left(h_xh_{xy}+h_yh_{yy}\right)}{\sqrt{1+h_x^2+h_y^2}^7}$$

$$+15\frac{h_{yy}\left(h_xh_{xx}+h_yh_{yx}\right)^2}{\sqrt{1+h_x^2+h_y^2}^7}-30\frac{h_{xy}\left(h_xh_{xy}+h_yh_{yy}\right)\left(h_xh_{xx}+h_yh_{yx}\right)}{\sqrt{1+h_x^2+h_y^2}^7}$$

$$+15\frac{h_{xx}\left(h_xh_{xy}+h_yh_{yy}\right)^2}{\sqrt{1+h_x^2+h_y^2}^7}.$$

All of the fractions on the right-hand side have common denominators and so we can combine them into one big fraction. The *numerator* of this fraction will be:

$$-15\left(h_{xx}h_{yy}-h_{xy}^2\right)h_x\left(h_xh_{xx}+h_yh_{yx}\right)-15\left(h_{xx}h_{yy}-h_{xy}^2\right)h_y\left(h_xh_{xy}+h_yh_{yy}\right)$$

$$+15h_{yy}\left(h_xh_{xx}+h_yh_{yx}\right)^2-30h_{xy}\left(h_xh_{xy}+h_yh_{yy}\right)\left(h_xh_{xx}+h_yh_{yx}\right)$$

$$+15h_{xx}\left(h_xh_{xy}+h_yh_{yy}\right)^2.$$

Expanding this out gives that this numerator is equal to:

$$-15h_{xx}^2h_{yy}h_x^2-15h_{xx}h_{yy}h_{xy}h_xh_y+15h_{xx}h_{xy}^2h_x^2+15h_{xy}^3h_xh_y-15h_{xx}h_{yy}h_{xy}h_xh_y$$

$$-15h_{xx}h_{yy}^2h_y^2+15h_{xy}^3h_xh_y+15h_{xy}^2h_{yy}h_y^2+15h_x^2h_{xx}^2h_{yy}+30h_{xx}h_{xy}h_{yy}h_xh_y$$

$$+15h_{xy}^2h_{yy}h_y^2-30h_{xx}h_{xy}^2h_x^2-30h_{xx}h_{xy}h_{yy}h_xh_y-30h_{xy}^3h_xh_y-30h_{yy}h_{xy}^2h_y^2$$

$$+15h_x^2h_{xx}h_{xy}^2+30h_{xx}h_{xy}h_{yy}h_xh_y+15h_{xx}h_{yy}^2h_y^2.$$

Notice that all of the terms in the above expression cancel out. So the numerator of the fraction equal to $\frac{\delta J}{\delta h}[h]$ is equal to zero and thus finally we get that:

$$\frac{\delta J}{\delta h}[h]=0.$$

Interesting, we get that the functional derivative of J is equal to zero at *any* h in the domain of the functional.[38] In particular, the functional derivative of J at Λ for any time $t\in[0,a]$ is equal to zero:

[38] Just as an interesting note, by a not too hard generalization of Theorem 3.6.5 that is direct corollary of Theorem 3.7.1, this in fact means that J is constant on its whole domain.

$$\frac{\delta J}{\delta h}[\Lambda] \equiv 0.$$

Plugging this into our last expression above for the time derivative of total Gaussian curvature $K[\mathcal{S}(t)]$ gives us that:

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \iint_{\overline{B_{3r/4}(p_{xy})}} \frac{\delta J}{\delta h}[\Lambda]\Lambda_t dxdy = 0.$$

So, the total Gaussian curvature $K[\mathcal{S}(t)]$ is constant for all time $t \in [0, a]$ and is equal to $K[\mathcal{S}(0)] = K[S]$. With this we have proven the theorem.

■

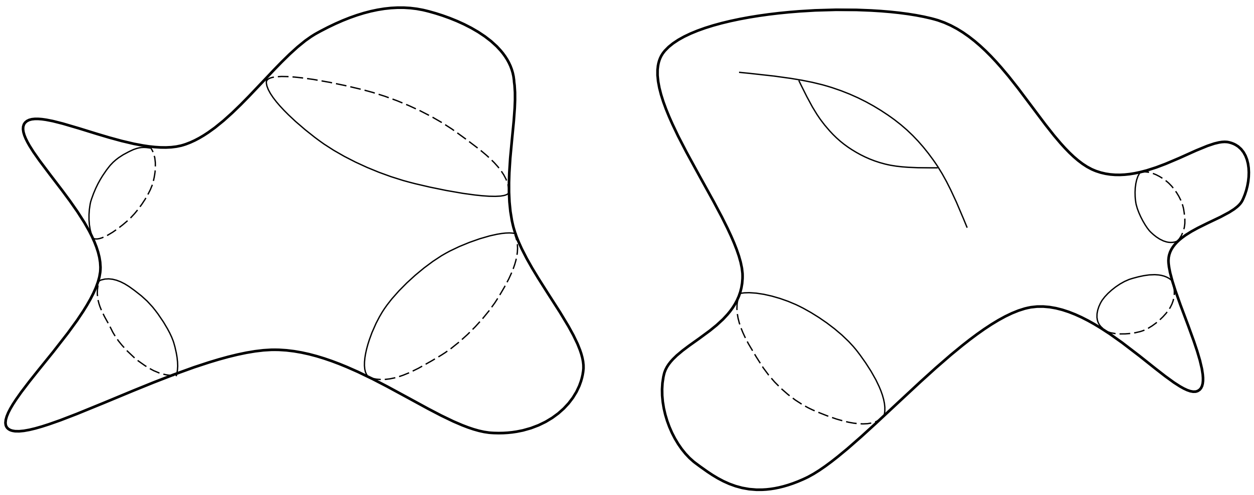
In the above proof we showed that the total Gaussian curvature is invariant under smooth local graph deformations and at the heart of the proof was to show that the functional derivative of the "local total Gaussian curvature functional" described in the above proof:

$$J[h] = \iint_{\Omega} \frac{h_{xx}h_{yy} - h_{xy}^2}{\sqrt{1 + h_x^2 + h_y^2}^3} dxdy$$

is constantly zero everywhere (here I set $\Omega = \overline{B_{3r/4}(p_{xy})}$). The fact that the functional derivative of the above functional is equal to zero is in fact the whole idea behind why the total Gaussian curvature is invariant under such local deformations. The reason for this is that in essence the above functional describes the total Gaussian curvature of a small local piece of the surface that is being deformed a little bit. By showing that its variational derivative is equal to zero you show that the total Gaussian curvature over this small piece is unchanging under any such deformation and thus the total Gaussian curvature of the whole surface is invariant under any deformation that deforms the surface locally at any of its points.

I also do want to mention that the calculation involved in the above proof when we calculated the functional derivative of the functional J was rather long and intimidating looking. Even though there is nothing more wonderful than to pass over and get lost in the realm of mathematics while doing such long calculations, there is a trick that involves rotating the surfaces in the domain of the functional in order to evaluate the functional's derivative pointwise that helps shorten the above calculation around 8-fold. We will discuss this trick in the next chapter and I will show you how it can be used to do the above calculation much simpler. We will then use this trick in the proof of the n-dimensional version of the above theorem because there it will be nearly impossible to do the calculation without this trick of rotating the surfaces. In fact, you may skip to Section 6 of Chapter 6 right now if you wish to learn how to use this technique to shorten the above mentioned calculation without loss of continuity.

So what does the above theorem give us? It tells us that if we can obtain a surface by multiple applications of smooth local graph deformation to one of the fundamental surfaces, then the total Gaussian curvature of your surface will be the same as that fundamental surface that you obtained it from. Some examples of such surfaces can look like:



The surface on the left was obtained by doing a lot of local smooth graph deformations to the sphere while the surface on the right was obtained by doing a lot of local smooth graph deformations to the torus. By the above theorem, since the total Gaussian curvature of a surface is unchanging under each such deformation, we get that the integral of the Gaussian curvature over the whole surface on the left is 4π (the total Gaussian curvature of a sphere - we will show this below) and the integral of the Gaussian curvature over the whole surface on the right is 0 (the total Gaussian curvature of a torus). It's amazing that you can just look at such a surface and visually tell what is the integral of the Gaussian curvature over the whole surface. It's all due to how the Gaussian curvature redistributes itself under such smooth surface deformations.

The total Gaussian curvature of the sphere and torus can be computed directly. Let's do the sphere case and I will leave the calculation of the total Gaussian curvature of the torus to the reader.

Note 5.6.3: Let's compute the total Gaussian curvature of the sphere of radius $r > 0$. At the end of Chapter 4 Section 4 we derived that the Gaussian curvature of a sphere of radius $r > 0$ at any point is equal to $1/r^2$ (see Example 4.4.7). So, the total Gaussian curvature of this sphere is given by:

$$K[\partial B_r(0)] = \iint\limits_{\partial B_r(0)} K \, d\sigma = \iint\limits_{\partial B_r(0)} \frac{1}{r^2} d\sigma = \frac{1}{r^2} \iint\limits_{\partial B_r(0)} d\sigma = \frac{1}{r^2} 4\pi r^2 = 4\pi.$$

And so indeed the total Gaussian curvature of a sphere of any radius $r > 0$ is 4π.

Chapter 6: Variational Differential Geometry in $\mathbb{R}^n$

"Oh calculus, thy beauty surpasses most of what I see.

Think about you I do day and night,

Though torture me you do on Thursday night." – Student

Section 1: Outline

In this chapter we will be doing all of the differential geometry that we have done so far but in higher dimensions. The generalization of the differential geometry that we have done so far to higher dimensions is actually quite natural and not too difficult since nothing that we have done so far was characteristic to the fact that we were working with two-dimensional surfaces or that we were working in three-dimensional space. For example, most of the foundational definitions and theorems that we had in the beginning portion of Chapter 4 can easily be restated in higher dimensions and their proofs are pretty much exactly the same as in the three-dimensional case except for that you have to keep track of more variables.

In the next section we will state all of the foundational definitions and theorems for n-dimensional differential geometry and from there we will go ahead and prove the higher dimensional versions of the three main theorems that we proved in the previous chapter: the Minimizing Curve Theorem, the Minimal Surface Theorem, and a version of a corollary of the Global Gauss-Bonnet Theorem. We will end the chapter with an introduction to a more general manifold theory (including a proof of the general Minimizing Curve Theorem) and a possible proof of a higher dimensional version of Theorema Egregium [see future edition of this book].

Section 2: Fundamental Definitions in n-Dimensional Differential Geometry

Note: In this chapter, the word **plane** in $\mathbb{R}^n$ will mean an $(n-1)$-dimensional linear subspace. A lot of people use the word hyperplane for this.

First order of business is of course to define what a surface in higher dimensional space is. More precisely, lets define what a surface sitting in $\mathbb{R}^n$ for any general $n \in \mathbb{Z}_+$ is. In three-dimensional space, a surface colloquially speaking was a curved version of the Euclidean space that had dimension one smaller than 3 sitting in $\mathbb{R}^3$. In other words, we had that a surface in three-dimensional space was a curved version of $\mathbb{R}^2$ sitting in $\mathbb{R}^3$. Same thing is done in higher dimensional space. A surface in $\mathbb{R}^n$ colloquially speaking is a curved version of $\mathbb{R}^{n-1}$ that is sitting in $\mathbb{R}^n$.[39]

One convenient way to represent surfaces sitting in $\mathbb{R}^n$ is, just like in $\mathbb{R}^3$, to represent them as graphs of functions of the form $f(x_1, x_2, \dots, x_{n-1})$ (note that there are one less amount of variable of f than the dimension of the space). Indeed, this leads us to our first definition of a surface.

Definition 6.2.1 (First Definition of a Smooth $(\boldsymbol{n-1})$-Dimensional Surface Sitting in $\mathbb{R}^{\boldsymbol{n}}$): *A set $S \subseteq \mathbb{R}^n$ is called a **smooth $(\boldsymbol{n-1})$-dimensional surface** sitting in $\mathbb{R}^n$ if it satisfies the following property: for any point $p \in S$, locally to p S is the graph of a $C^\infty[\mathbb{R}^{n-1}]$ function over some open set. Stated more precisely, S is a smooth $(n-1)$-dimensional surface if for any $p \in S$ there exists an open set V and a function of the form:*

$$x_n = f(x_1, x_2, \dots, x_{n-1}), \qquad x_{n-1} = f(x_1, x_2, \dots, x_{n-2}, x_n), \dots,$$

$$x_k = f(x_1, x_2, \dots x_{k-1}, x_{k+1}, \dots x_n), \qquad \dots, \qquad \text{or} \quad x_1 = f(x_2, x_3, \dots, x_n)$$

(in other words, one variable is a function of the rest) where $f \in C^\infty[U]$ and U is some open set in $\mathbb{R}^{n-1}$ such that $V \cap S$ is the graph of f over U.

This is the analog of Definition 4.2.1 that we had in Chapter 4 in which we define smooth surfaces as graph of infinitely differentiable functions. The other equivalent definition of a smooth $(n-1)$-dimensional surface in $\mathbb{R}^n$ uses surface parametrizations.

Definition 6.2.2 (Second Definition of a Smooth $(\boldsymbol{n-1})$-Dimensional Surface Sitting in $\mathbb{R}^{\boldsymbol{n}}$): *A set $S \subseteq \mathbb{R}^3$ is called a **smooth $(\boldsymbol{n-1})$-dimensional surface** sitting in $\mathbb{R}^n$ if for every point $p \in S$ on the surface, there exists a surface parametrization of S located at p. A **surface parametrization** of a surface S is a function of the form $\Phi : U \subseteq \mathbb{R}^{n-1} \to V \cap S$ where U and V are open sets that satisfies the following three properties:*

[39] Some people like to refer to surfaces sitting in Euclidean space of dimensions higher than three as "hypersurfaces," just like we colloquially did in previous chapters when we talked about functions of the form $h(x_1, x_2, \dots, x_m)$. However here I am not going to stick to such a convention since there is really nothing special about surfaces sitting in three-dimensional space and so I see no need for a special term to refer to them.

1.)

$$\Phi(u_1, u_2, \dots, u_{n-1}) \\ = \big(\Phi_1(u_1, u_2, \dots, u_{n-1}), \Phi_2(u_1, u_2, \dots, u_{n-1}), \dots, \Phi_n(u_1, u_2, \dots, u_{n-1})\big)$$

is infinitely differentiable. Symbolically this is written as $\Phi \in C^\infty[U]$ *or more precisely* $\Phi_1, \Phi_2, \dots, \Phi_n \in C^\infty[U]$.

2.) Φ *is a homeomorphism.*

3.) The differential of Φ *has maximal rank everywhere. In other words, for every point* $(u_1, u_2, \dots, u_{n-1}) \in U$ *in the domain of* Φ*, the differential matrix of* Φ*:*

$$D\Phi(u_1, u_2, \dots, u_{n-1})$$

$$= \begin{bmatrix} \frac{\partial \Phi_1}{\partial u_1}(u_1, u_2, \dots, u_{n-1}) & \frac{\partial \Phi_1}{\partial u_2}(u_1, u_2, \dots, u_{n-1}) & \cdots & \frac{\partial \Phi_1}{\partial u_n}(u_1, u_2, \dots, u_{n-1}) \\ \frac{\partial \Phi_2}{\partial u_1}(u_1, u_2, \dots, u_{n-1}) & \frac{\partial \Phi_2}{\partial u_2}(u_1, u_2, \dots, u_{n-1}) & \cdots & \frac{\partial \Phi_2}{\partial u_n}(u_1, u_2, \dots, u_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Phi_n}{\partial u_1}(u_1, u_2, \dots, u_{n-1}) & \frac{\partial \Phi_n}{\partial u_2}(u_1, u_2, \dots, u_{n-1}) & \cdots & \frac{\partial \Phi_n}{\partial u_n}(u_1, u_2, \dots, u_{n-1}) \end{bmatrix}$$

has maximal rank ("maximal rank" means that some $(n-1)$ *by* $(n-1)$ *submatrix of the above* n *by* $(n-1)$ *matrix has non-zero determinant). An equivalent way to state this is that the following Jacobians:*

$$\frac{\partial(\Phi_1, \Phi_2, \dots, \Phi_{n-1})}{\partial(u_1, u_2, \dots, u_{n-1})}, \qquad \frac{\partial(\Phi_1, \Phi_2, \dots, \Phi_{n-2}, \Phi_n)}{\partial(u_1, u_2, \dots, u_{n-1})}, \qquad \dots,$$

$$\frac{\partial(\Phi_1, \Phi_2, \dots, \Phi_{k-1}, \Phi_{k+1}, \dots, \Phi_n)}{\partial(u_1, u_2, \dots, u_{n-1})}, \qquad \dots, \qquad \frac{\partial(\Phi_2, \Phi_3, \dots, \Phi_n)}{\partial(u_1, u_2, \dots, u_{n-1})}$$

don't ever vanish simultaneously on U *(these are in fact the* $(n-1)$ *by* $(n-1)$ *submatrices of the above* n *by* $(n-1)$ *matrix).*

Now we say that the surface parametrization Φ *is "located at* $p \in S$*" (or simply "at* p*") if* $p \in ran(\Phi)$*. So as already stated above,* S *is a surface if at every point* $p \in S$ *on the surface there exists a surface parametrization* Φ *of* S *located at* p*.*

The proof of the equivalence of the above two definitions of a smooth $(n-1)$-dimensional surface sitting in $\mathbb{R}^n$ is very similar to the proof of Theorem 4.2.8 and is left to the reader. You might notice that there is no convenient cross product reformulation of the third condition of a surface parametrization in the above definition. This is due to the fact that we don't have a cross product in $\mathbb{R}^n$. A rather unfortunate fact because in three-dimensional differential geometry the cross product was a very convenient tool to for example construct vectors that were perpendicular to surfaces (which we needed to construct the Gauss map). In $\mathbb{R}^n$ we can't do this this using the cross product and so we need to find a more direct way of constructing such perpendicular vectors to surfaces. However, before we can even talk about vectors being perpendicular to surfaces we need to define the tangent plane to a general surface since orthogonality is always defined as a relation to a linear space.

Definition 6.2.3: *Let S be a smooth $(n-1)$-dimensional surface sitting in $\mathbb{R}^n$ and let $p \in S$ be any point on this surface. Let Γ be the set of all C^∞ curves of the form $\gamma : [-1,1] \to \mathbb{R}^n$ that lie on the surface S ("lying on the surface" means that for any $t \in [-1,1]$, $\gamma(t) \in S$) and that pass through the point p at time $t = 0$: $p = \gamma(0)$. Let $T_p(s)$ be the set:*

$$T_p(s) = \{\gamma'(0) : \gamma \in \Gamma\}.$$

*It turns out that $T_p(s)$ is a plane in $\mathbb{R}^3$ and it is called the **tangent plane** to S at the point p. Furthermore, if Φ is a surface parametrization of S at p and $\left(u_{1_0}, u_{2_0}, \dots, u_{(n-1)_0}\right) = \Phi^{-1}(p)$, then the plane $T_p(S)$ is spanned by all of the first partials of Φ at this point:*

$$T_p(S)$$
$$= span\left\{\frac{\partial\Phi}{\partial u_1}\left(u_{1_0}, u_{2_0}, \dots, u_{(n-1)_0}\right), \frac{\partial\Phi}{\partial u_2}\left(u_{1_0}, u_{2_0}, \dots, u_{(n-1)_0}\right), \dots, \frac{\partial\Phi}{\partial u_n}\left(u_{1_0}, u_{2_0}, \dots, u_{(n-1)_0}\right)\right\}.$$

The proof of the fact that $T_p(S)$ is indeed a plane is analogous to the proof of Theorem 4.2.11 in which you will need a higher dimensional version of Lemma 4.2.10.

Lemma 6.2.4: *Let S be a smooth $(n-1)$-dimensional surface sitting in $\mathbb{R}^n$ and let $\Phi : U \to V \cap S$ be a surface parametrization on this surface where the sets U and V are open in $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$ respectively. Suppose that $\gamma : [a,b] \to \mathbb{R}^n$ is a C^∞ curve that lies on the surface and is entirely in V (in other words, $\gamma(t)$ lies entirely in the image of Φ). Then the inverse image of this curve under the surface parametrization Φ:*

$$u(t) = \Phi^{-1}\big(\gamma(t)\big)$$

is also a C^∞ curve lying in the domain of Φ.

Details and the proof of the above lemma, which is not a difficult generalization of the arguments in the proof of Lemma 4.2.10, are left to the reader. The fact that the vectors $\left\{\frac{\partial\Phi}{\partial u_1}, \frac{\partial\Phi}{\partial u_2}, \dots, \frac{\partial\Phi}{\partial u_n}\right\}$ are linearly independent comes from the fact that the differential matrix $D\Phi$ always has maximum rank (see the third condition of a surface parametrization in Definition 6.2.2).

Now that we have the concept of a tangent plane to a general surface sitting in $\mathbb{R}^n$, we can now define what it means for a vector to be perpendicular to a surface. As we did before we simply say that a vector N is perpendicular to a surface at a point if it is perpendicular to the tangent plane at that point.

Definition 6.2.5: *Let S be a smooth $(n-1)$-dimensional surface sitting in $\mathbb{R}^n$ and $p \in S$ be any point on it. Then N is said to be "**normal**" or "**perpendicular**" to the surface S at the point p if it is perpendicular to the tangent plane $T_p(S)$ to the surface at the point p.*

The difficulty that we now face is: how do we write an explicit equation for a normal vector to the surface given a surface parametrization. This is not so easy as before since now we are working in $\mathbb{R}^n$ and thus have no access to a vector cross product that could give us such a vector explicitly. In fact, a more crucial question is how do we prove that in a neighborhood of any point of a surface there exists a smooth function that each point of the surface in its domain gives a unit normal vector to the surface. Such functions are called Gauss maps and as before we need them to be able to define the curvatures of a surface.

Let's first solve this problem when our surface S is the graph of a function $f \in C^\infty(U)$ where U is some open set in $\mathbb{R}^{n-1}$. Suppose that our function f is of the form:

$$x_n = f(x_1, x_2, \dots, x_{n-1})$$

and that:

$$p = \left(u_{1_0}, u_{2_0}, \dots, u_{(n-1)_0}, f\left(u_{1_0}, u_{2_0}, \dots, u_{(n-1)_0}\right)\right)$$

is some point on the surface S (which means that $\left(u_{1_0}, u_{2_0}, \dots, u_{(n-1)_0}\right) \in U$).[40] Now, we want to find an explicit equation for a vector that is normal to the surface at the point p. For brevity's sake, in the following let us set:

$$u_0 = \left(u_{1_0}, u_{2_0}, \dots, u_{(n-1)_0}\right)$$

$$x = (x_1, x_2, \dots, x_{n-1})$$

so that we can write the arguments of f more compactly. There are many ways of going about finding such a normal vector and probably the most standard trick is to first notice that S is the zero level set of the function $g(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_{n-1}) - x_n$:

$$S = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : g(x_1, x_2, \dots, x_n) = 0\}.$$

Using the fact from multivariable calculus that ∇g is always perpendicular to any of its level set, we get that the vector:

$$\left(\frac{\partial f}{\partial x_1}(u_0), \frac{\partial f}{\partial x_2}(u_0), \dots, \frac{\partial f}{\partial x_{n-1}}(u_0), -1\right)$$

is perpendicular to the surface at the point p. This can also be seen directly. Indeed, take any C^∞ curve $\gamma(t)$ that lies on the surface S and that passes through p at time $t = 0$. $\gamma(t)$ has an explicit form:

[40] In the other cases of the forms of f mentioned in Definition 6.2.1 the following analysis is similar.

$$\gamma(t) = \Big(u_1(t), u_2(t), \dots, u_2(t), f\big(u(t)\big)\Big)$$

where we set:

$$u(t) = \big(u_1(t), u_2(t), \dots, u_2(t)\big)$$

to be some curve in U. Then we have that:

$$\left(\frac{\partial f}{\partial x_1}(u_0), \frac{\partial f}{\partial x_2}(u_0), \dots, \frac{\partial f}{\partial x_{n-1}}(u_0), -1\right) \cdot \gamma'(0)$$

$$= \left(\frac{\partial f}{\partial x_1}(u_0), \frac{\partial f}{\partial x_2}(u_0), \dots, \frac{\partial f}{\partial x_{n-1}}(u_0), -1\right) \cdot \Big(u_1'(0), u_2'(0), \dots, u_2'(0), \nabla f\big(u(0)\big) \cdot \nabla u(0)\Big)$$

$$= \nabla f\big(u(0)\big) \cdot \nabla u(0) - \nabla f\big(u(0)\big) \cdot \nabla u(0) = 0$$

and thus the vector $\left(\frac{\partial f}{\partial x_1}(u_0), \frac{\partial f}{\partial x_2}(u_0), \dots, \frac{\partial f}{\partial x_{n-1}}(u_0), -1\right) \perp \gamma'(0)$. Since by definition any vector in $T_p(S)$ is of the form $\gamma'(0)$ for some C^∞ surface curve $\gamma(t)$, the above shows that this vector is indeed perpendicular to the surface S at p. Actually, the above shows more. If we set:

$$N(x) = \frac{\left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_{n-1}}(x), -1\right)}{\left\|\left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_{n-1}}(x), -1\right)\right\|},$$

or more explicitly:

Equation 6.2.6:
$$N(x) = \frac{\left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_{n-1}}(x), -1\right)}{\sqrt{1 + \sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}(x)\right)^2}},$$

then we get that N is a C^∞ function that gives a unit normal vector to the surface in a neighborhood of p, or more precisely in a neighborhood of its projection onto the x_1-x_2…-x_{n-1} plane: u_0 (since technically the domain of N is a subset of the x_1-x_2…-x_{n-1} plane). We don't have the trouble that the above denominator is ever zero since it is always bigger than or equal to 1. This function N is important because in the graph of f surface parametrization of S, N is the Gauss map near p. This can now be used to prove the existence of the Gauss map near any point in a surface in any type of surface parametrization. Let's do this in the following theorem.

Theorem 6.2.7: *Suppose that S is a smooth $(n-1)$-dimensional surface sitting in $\mathbb{R}^n$ and that $p \in S$ is some point on it. Suppose also that $\Phi : U \to V \cap S$ is a surface parametrization of S at p. Then, there exists an open subset $W \subseteq U$ that contains $\Phi^{-1}(p)$ and a function $N \in C^\infty[W, \mathbb{R}^n]$ such that for any $(u_1, u_2, \dots, u_{n-1}) \in U$, the vector*

$$N(u_1, u_2, \dots, u_{n-1})$$

is a unit normal vector to the surface at $\Phi(u_1, u_2, \dots, u_{n-1})$*. Such a function* N *is called a* ***Gauss map***.

Proof: We already showed the existence of the Gauss map when the surface parametrization is a graph surface parametrization in the above discussion. We can in fact use that special case to prove the more general result stated in this theorem. In this proof, let u denote the vector:

$$u = (u_1, u_2, \dots, u_{n-1}).$$

Since S is a smooth surface, there exist open sets $\mathcal{U} \subseteq \mathbb{R}^{n-1}$ and $\mathcal{J} \subseteq \mathbb{R}^n$ and a function $f \in C^\infty(\mathcal{U})$ such that $p \in \mathcal{J}$ and $\mathcal{J} \cap S$ is the graph of f over $\mathcal{U}$. Let us suppose that f is of the form:

$$x_n = f(x_1, x_2, \dots, x_{n-1}).$$

This proof in the other cases of the forms of f mentioned in Definition 6.2.1 is pretty much exactly the same. Notice that $V \cap \mathcal{J}$ is an open neighborhood of p. And $\Phi^{-1}[V \cap \mathcal{J}]$ is an open neighborhood of $\Phi^{-1}(p)$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection map from $\mathbb{R}^n$ onto the x_1-x_2-…-x_{n-1} plane. Now, define the map $N : \Phi^{-1}[V \cap \mathcal{J}] \to \mathbb{R}^n$ simply by:

$$N(u) = \frac{\left(\frac{\partial f}{\partial x_1}\big((\pi \circ \Phi)(u)\big), \frac{\partial f}{\partial x_2}\big((\pi \circ \Phi)(u)\big), \dots, \frac{\partial f}{\partial x_{n-1}}\big((\pi \circ \Phi)(u)\big), -1\right)}{\sqrt{1 + \sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}\big((\pi \circ \Phi)(u)\big)\right)^2}}.$$

Or more concisely:

$$\mathit{Equation}\ 6.2.8\text{:}\quad N = \frac{\left(\frac{\partial f}{\partial x_1}(\pi \circ \Phi), \frac{\partial f}{\partial x_2}(\pi \circ \Phi), \dots, \frac{\partial f}{\partial x_{n-1}}(\pi \circ \Phi), -1\right)}{\sqrt{1 + \sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}(\pi \circ \Phi)\right)^2}}.$$

I claim that this is a Gauss map. To prove that this is a Gauss map, and thus the map that want, notice that by the discussion before this theorem for each u in the domain of N, $N(u)$ is a unit normal vector to the surface at the point (here I use the fact that $\pi \circ \Phi = (\Phi_1, \Phi_2, \dots, \Phi_{n-1})$):

$$\Big(\Phi_1(u), \Phi_2(u), \dots, \Phi_{n-1}(u), f\big((\pi \circ \Phi)(u)\big)\Big) = \big(\Phi_1(u), \Phi_2(u), \dots, \Phi_n(u)\big) = \Phi(u).$$

And this function N is C^∞ since it is the composition of many C^∞ functions and a division by a nonzero C^∞ function ($\pi \circ \Phi$ is C^∞ since $\pi \circ \Phi = (\Phi_1, \Phi_2, \dots, \Phi_{n-1})$). So N is the map that we want and thus is a Gauss map.

■

When one first reads Equation 6.2.8 above they might wonder "where did that come from?" or "how did you know to do that?" The reason for that is actually pretty simple: you just have to look at what the right-hand side does procedurally. Notice that all the right-hand side does is it takes a point u in the domain of the surface parametrization Φ, takes it to the point $\Phi(u)$ on the surface, projects it into the domain plane of f (the x_1-x_2-…-x_{n-1} plane in this case), and then uses the formula that we derived in the discussion before the above theorem (Equation 6.2.6) to get a perpendicular vector to the surface right above that projected point. If you mathematically write this procedure out, you will then write Equation 6.2.6 composed with π composed with Φ. This is exactly what Equation 6.2.8 is.

The existence of a Gauss map near any point on a surface in any surface parametrization is important because it will be the crucial ingredient, as it was in Chapter 4, to define the curvatures of a general surface sitting in $\mathbb{R}^n$. This we will do in the Section 4 below.

Section 3: Level Set Representations of Surfaces

A very important technique in variational differential geometry is to represent surfaces locally as the level set of nonsingular functions. In the previous chapter we saw that this technique applied to a surface sitting in $\mathbb{R}^3$ allowed us to use the surface Euler-Lagrange vector differential equation on the arclength functional in order to derive the minimizing curve differential equation. The same exact thing goes for the case of surfaces sitting in general Euclidean space $\mathbb{R}^n$ and so in order to use the hypersurface Euler-Lagrange vector differential equation to derive the minimize curve differential equation for general surfaces we will need to be able to represent them as level sets of nonsingular functions.

As with surfaces sitting in $\mathbb{R}^3$, the level set of smooth nonsingular functions are smooth regular surfaces in $\mathbb{R}^n$.

Theorem 6.3.1: *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function (meaning that $g \in C^\infty[\mathbb{R}^n]$). Let S be the level set of g:*

$$S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = c\}$$

where $c \in \mathbb{R}$ is some fixed constant. Suppose also that ∇g never vanishes on S. Then S is a smooth surface.

This is proved exactly like Theorem 4.3.1 by a direct application of the implicit function theorem except that here more variables are more involved. However to us a more important theorem is the following because it is what is going to allow us to apply the hypersurface Euler-Lagrange vector differential equation to arclength functionals on surfaces.

Theorem 6.3.2: *Suppose that S is a smooth $(n-1)$-dimensional surface sitting in $\mathbb{R}^n$. Then locally to any point $p \in S$, S can be represented as the level set of a C^∞ function. In other words, for any point $p \in S$ there exists an open set $V \subseteq \mathbb{R}^n$ that contains p and $V \cap S$ is the level set of a C^∞ function $g(x_1, x_2, \dots, x_n)$ such that the partials $\partial g / \partial x_k$ for $k \in \{1,2, \dots, n\}$ never vanish simultaneously on $V \cap S$ (in other words, ∇g is never zero on $V \cap S$).*

The above theorem is proved by first considering the fact that in a neighborhood of any point p on a surface S, the surface can be represented as the graph of a function. Let's suppose that the surface near that point is the graph of a function of the form:

$$x_n = f(x_1, x_2, \dots, x_{n-1}).$$

Then locally near that point the surface is the level set of the function:

$$g(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_{n-1}) - x_n.$$

In the other cases of the forms of f mentioned in Definition 6.2.1 the construction of such a g is similar.

And as before, a magical tool of obtaining a normal vector to a level set of a nonsingular function g is to just take the gradient of the function: ∇g. This is a standard fact that is taught in many calculus courses and is proved similarly to how we proved it in the case of level sets of three variables functions g in the discussion after Theorem 4.3.2.

Section 4: Surface Curvatures, the Metric Tensor, and the Christoffel Symbols

Now that we have the existence of Gauss maps locally to any point on surfaces sitting in $\mathbb{R}^n$ in any surface parametrization, we are now ready to define the surface curvatures of such surfaces. Their definitions are just a direct generalization of Definition 4.4.5 of the Gaussian and mean curvature of surfaces sitting in $\mathbb{R}^3$.

Definition 6.4.1: *Let S be a smooth $(n-1)$-dimensional surface sitting in $\mathbb{R}^n$ and $p \in S$ be any point on it. By the definition of a smooth surface, we know that there exists some surface parametrization $\Phi : U \to V \cap S$ of S at p where $U \subseteq \mathbb{R}^{n-1}$ and $V \subseteq \mathbb{R}^n$ are open sets. Let $u_0 = \left(u_{1_0}, u_{2_0}, \dots, u_{n-1_0}\right) = \Phi^{-1}(p)$ and let u denote the vector:*

$$u = (u_1, u_2, \dots, u_n)$$

Let $N : W \to \mathbb{R}^n$ be a Gauss map for S where W is some open subset of U that contains u_0. The ***first fundamental form*** *of S is defined as:*

$$\forall k, j \in \{1,2, \dots, n-1\}, \qquad \beta_{k,j}(u) = \frac{\partial \Phi}{\partial u_k}(u) \cdot \frac{\partial \Phi}{\partial u_j}(u).$$

And the ***second fundamental form*** *of S is defined as:*

$$\forall k, j \in \{1,2, \dots, n-1\}, \qquad \psi_{k,j}(u) = \frac{\partial N}{\partial u_k}(u) \cdot \frac{\partial \Phi}{\partial u_j}(u)$$

wherever the Gauss map is defined (on $u \in W$ that is). Then the ***Gaussian and mean curvatures*** *of S at the point p are defined as:*

$$K(u_0) = \det\left(\begin{bmatrix} \psi_{1,1}(u_0) & \cdots & \psi_{1,n-1}(u_0) \\ \vdots & \ddots & \vdots \\ \psi_{n-1,1}(u_0) & \cdots & \psi_{n,n}(u_0) \end{bmatrix} \begin{bmatrix} \beta_{1,1}(u_0) & \cdots & \beta_{1,n-1}(u_0) \\ \vdots & \ddots & \vdots \\ \beta_{n-1,1}(u_0) & \cdots & \beta_{n-1,n-1}(u_0) \end{bmatrix}^{-1}\right),$$

$$H(u_0) = \frac{1}{n} trace\left(\begin{bmatrix} \psi_{1,1}(u_0) & \cdots & \psi_{1,n-1}(u_0) \\ \vdots & \ddots & \vdots \\ \psi_{n-1,1}(u_0) & \cdots & \psi_{n,n}(u_0) \end{bmatrix} \begin{bmatrix} \beta_{1,1}(u_0) & \cdots & \beta_{1,n-1}(u_0) \\ \vdots & \ddots & \vdots \\ \beta_{n-1,1}(u_0) & \cdots & \beta_{n-1,n-1}(u_0) \end{bmatrix}^{-1}\right).$$

As a point of interest, the second fundamental form is also given by:

$$\forall k, j \in \{1,2, \dots, n-1\}, \qquad \psi_{k,j}(u) = -N(u) \cdot \frac{\partial^2 \Phi}{\partial u_j \partial u_k}(u),$$

which are often easier to calculate than the above alternative equations for the second fundamental form (see below for proof).

The reason why the above two equations for the second fundamental are the same comes from the fact that N is orthogonal to every one of the vectors $\partial \Phi / \partial u_k$: for any $k, j \in \{1,2, \dots, n-1\}$,

$$0 = \frac{\partial}{\partial u_k}\left(N \cdot \frac{\partial \Phi}{\partial u_j}\right) = \frac{\partial N}{\partial u_k} \cdot \frac{\partial \Phi}{\partial u_j} + N \cdot \frac{\partial^2 \Phi}{\partial u_j \partial u_k}$$

and thus:

$$\frac{\partial N}{\partial u_k} \cdot \frac{\partial \Phi}{\partial u_j} = -N \cdot \frac{\partial^2 \Phi}{\partial u_j \partial u_k}.$$

The Gaussian and mean curvatures stem from exactly the same sort of geometric interpretation that we had for these curvatures of 2 dimensional surfaces in Section 4 of Chapter 4 before Definition 4.4.5 in higher dimensions. And such a discussion on general surfaces in $\mathbb{R}^n$ in fact shows that the Gaussian and mean curvatures in the above definition are well defined up to sign even if you use different surface parametrizations of the surface S and/or different Gauss map near the point p. In fact, it turns out that if the dimension of the surface is even then the value of the Gaussian curvature at your point will always be the same no matter what surface parametrization or Gauss map you use for the surface at your point in order to calculate it.

Example 6.4.2: Let us compute the Gaussian and mean curvatures of a surface sitting in $\mathbb{R}^n$ when it is parametrized by a graph surface parametrization. Let $\Phi : U \to \mathbb{R}^n \cap S$ be the surface parametrization, where $U \subseteq \mathbb{R}^{n-1}$ is an open set, given by:

$$\Phi(x_1, x_2, \dots, x_{n-1}) = \big(x_1, x_2, \dots, x_{n-1}, f(x_1, x_2, \dots, x_{n-1})\big)$$

where $f \in C^\infty[U]$. Here Φ is surface parametrization for the surface generated by the graph of the function f. Let x here denote the vector:

$$x = (x_1, x_2, \dots, x_n).$$

One amazing thing that we have here is that by the discussion in the previous section we can easily construct a Gauss map defined over all of U:

$$N(x) = \frac{\left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \ldots, \frac{\partial f}{\partial x_{n-1}}(x), -1\right)}{\sqrt{1 + \sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}(x)\right)^2}}.$$

Great! Now, if we compute the partials of Φ we will see that for any $k \in \{1,2, \ldots, n-1\}$,

$$\beta_{k,k}(x) = \frac{\partial \Phi}{\partial x_k}(x) \cdot \frac{\partial \Phi}{\partial x_k}(x) = 1 + \left(\frac{\partial f}{\partial x_k}(x)\right)^2.$$

And for any $k, j \in \{1,2, \ldots, n-1\} : k \neq j$,

$$\beta_{k,j}(x) = \frac{\partial \Phi}{\partial x_k}(x) \cdot \frac{\partial \Phi}{\partial x_j}(x) = \frac{\partial f}{\partial x_k}(x)\frac{\partial f}{\partial x_j}(x).$$

Furthermore, for any $k, j \in \{1,2, \ldots, n-1\}$,

$$\frac{\partial^2 \Phi}{\partial x_k \partial x_j}(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial^2 f}{\partial x_k \partial x_j}(x) \end{bmatrix}.$$

Taking the dot product of this vector with $N(x)$ gives that for any $k, j \in \{1,2, \ldots, n-1\}$,

$$\psi_{k,j} = -N(x) \cdot \frac{\partial^2 \Phi}{\partial x_k \partial x_j}(x) = \frac{\frac{\partial^2 f}{\partial x_k \partial x_j}(x)}{\sqrt{1 + \sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}(x)\right)^2}}.$$

Awesome! Having computed the first and second fundamental form we can now compute the Gaussian and mean curvatures of the surface. If we plug the above expressions into the matrices with their entries being the first and second fundamental forms respectively, you will get that:

$$\begin{bmatrix} \beta_{1,1}(u_0) & \cdots & \beta_{1,n-1}(u_0) \\ \vdots & \ddots & \vdots \\ \beta_{n-1,1}(u_0) & \cdots & \beta_{n-1,n-1}(u_0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \left(\frac{\partial f}{\partial x_1}(x)\right)^2 & \frac{\partial f}{\partial x_1}(x)\frac{\partial f}{\partial x_2}(x) & \cdots & \frac{\partial f}{\partial x_1}(x)\frac{\partial f}{\partial x_{n-1}}(x) \\ \frac{\partial f}{\partial x_2}(x)\frac{\partial f}{\partial x_1}(x) & 1 + \left(\frac{\partial f}{\partial x_2}(x)\right)^2 & \cdots & \frac{\partial f}{\partial x_2}(x)\frac{\partial f}{\partial x_{n-1}}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_{n-1}}(x)\frac{\partial f}{\partial x_1}(x) & \frac{\partial f}{\partial x_{n-1}}(x)\frac{\partial f}{\partial x_2}(x) & \cdots & 1 + \left(\frac{\partial f}{\partial x_{n-1}}(x)\right)^2 \end{bmatrix},$$

and:

$$\begin{bmatrix} \psi_{1,1}(x) & \cdots & \psi_{1,n-1}(x) \\ \vdots & \ddots & \vdots \\ \psi_{n-1,1}(x) & \cdots & \psi_{n,n}(x) \end{bmatrix}$$

$$= \frac{1}{\sqrt{1 + \sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}(x)\right)^2}} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_{n-1}} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_{n-1} \partial x_1} & \frac{\partial^2 f}{\partial x_{n-1} \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_{n-1}^2}(x) \end{bmatrix}.$$

Let us denote the first fundamental form matrix above by $M_\beta(x)$. And notice also that the matrix on the right-side of the expression for the second fundamental form matrix is the Hessian matrix $\mathcal{H}(x)$. Thus, the above equations can be neatly rewritten as:

$$\begin{bmatrix} \beta_{1,1}(u_0) & \cdots & \beta_{1,n-1}(u_0) \\ \vdots & \ddots & \vdots \\ \beta_{n-1,1}(u_0) & \cdots & \beta_{n-1,n-1}(u_0) \end{bmatrix} = M_\beta(x),$$

$$\begin{bmatrix} \psi_{1,1}(x) & \cdots & \psi_{1,n-1}(x) \\ \vdots & \ddots & \vdots \\ \psi_{n-1,1}(x) & \cdots & \psi_{n,n}(x) \end{bmatrix} = \frac{1}{\sqrt{1 + \sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}(x)\right)^2}} \mathcal{H}(x).$$

Plugging the above two equations into the equation for the Gaussian curvature in Definition 6.4.1 gives us that:

$$K(x) = \det\left(\frac{1}{\sqrt{1+\sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}(x)\right)^2}}\mathcal{H}(x)\left(M_\beta(x)\right)^{-1}\right)$$

$$= \frac{\det\left(\mathcal{H}(x)\right)}{\det\left(M_\beta(x)\right)\sqrt{1+\sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}(x)\right)^2}^{\,n-1}}$$

By the Cauchy-Binet equation (see Appendix B [see future edition of this book]), it is not hard to see that:

$$\det\left(M_\beta(x)\right) = 1 + \sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}(x)\right)^2,$$

and thus the above equation for the Gaussian curvature becomes:

$$K(x) = \frac{\det\left(\mathcal{H}(x)\right)}{\sqrt{1+\sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}(x)\right)^2}^{\,n+1}}$$

Plugging the above two equations for the first and second fundamental form matrices into the equation for the mean curvature in Definition 6.4.1 gives us that:

$$H(x) = \frac{1}{n}\,trace\left(\frac{1}{\sqrt{1+\sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}(x)\right)^2}}\mathcal{H}(x)\left(M_\beta(x)\right)^{-1}\right).$$

So we get that the Gaussian and mean curvatures of our surface are given by (here I omitted writing arguments):

$$K = \frac{\det(\mathcal{H})}{\sqrt{1+\sum_{k=1}^{n-1}\left(\frac{\partial f}{\partial x_k}(x)\right)^2}^{\,n+1}},$$

$$H = \frac{1}{n} trace\left(\frac{1}{\sqrt{1 + \sum_{k=1}^{n-1} \left(\frac{\partial f}{\partial x_k}\right)^2}} \mathcal{H} M_\beta^{-1} \right).$$

Similarly to what we had in Chapter 4, the matrix:

$$\begin{bmatrix} \beta_{1,1}(u_0) & \cdots & \beta_{1,n-1}(u_0) \\ \vdots & \ddots & \vdots \\ \beta_{n-1,1}(u_0) & \cdots & \beta_{n-1,n-1}(u_0) \end{bmatrix}$$

with its entries being the first fundamental form is called the **metric tensor** because it locally describes the metric of the surface in a surface parametrization. Indeed, to see this suppose that Φ is a surface parametrization of a surface S and that $\gamma : [a, b] \to \mathbb{R}^n$ is a C^∞ curve lying on the surface that is constantly inside of the range of Φ. Let:

$$u(t) = \big(u_1(t), u_2(t), \dots, u_{n-1}(t)\big) = \Phi^{-1}\big(\gamma(t)\big)$$

be the inverse image of $\gamma(t)$ under Φ ($u(t)$ is C^∞ by Lemma 6.2.4). Then the arclength of the surface curve $\gamma(t)$ from time $t = a$ to time $t = b$ is given by:

$$L[\gamma] = \int_a^b \|\gamma'(t)\| dt = \int_a^b \sqrt{\gamma'(t) \cdot \gamma'(t)} dt = \int_a^b \sqrt{\frac{d}{dt}\Big(\Phi\big(u(t)\big)\Big) \cdot \frac{d}{dt}\Big(\Phi\big(u(t)\big)\Big)}\, dt$$

$$= \int_a^b \sqrt{\left(\sum_{k=1}^{n-1} \frac{\partial \Phi}{\partial u_k}\big(u(t)\big) u_k'(t)\right) \cdot \left(\sum_{k=1}^{n-1} \frac{\partial \Phi}{\partial u_k}\big(u(t)\big) u_k'(t)\right)}\, dt$$

$$= \int_a^b \sqrt{\sum_{k=1}^{n-1} \sum_{j=1}^{n-1} \left(\frac{\partial \Phi}{\partial u_k}\big(u(t)\big) \cdot \frac{\partial \Phi}{\partial u_j}\big(u(t)\big)\right) u_k'(t) u_j'(t)}\, dt$$

$$= \int_a^b \sqrt{\left\langle \begin{bmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_{n-1}'(t) \end{bmatrix}, \begin{bmatrix} \beta_{1,1}(u_0) & \cdots & \beta_{1,n-1}(u_0) \\ \vdots & \ddots & \vdots \\ \beta_{n-1,1}(u_0) & \cdots & \beta_{n-1,n-1}(u_0) \end{bmatrix} \begin{bmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_{n-1}'(t) \end{bmatrix} \right\rangle}\, dt,$$

and so:

$$L[\gamma] = \int_a^b \sqrt{\left\langle \begin{bmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_{n-1}'(t) \end{bmatrix}, \begin{bmatrix} \beta_{1,1}(u_0) & \cdots & \beta_{1,n-1}(u_0) \\ \vdots & \ddots & \vdots \\ \beta_{n-1,1}(u_0) & \cdots & \beta_{n-1,n-1}(u_0) \end{bmatrix} \begin{bmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_{n-1}'(t) \end{bmatrix} \right\rangle} \, dt$$

where $\langle \quad \rangle$ denotes the standard Euclidian inner product. Thus, the arclength equation on a differential level can then be written as (using the differential formula $\frac{du_k}{dt} dt = du_k$):

$$dL^2 = \left\langle \begin{bmatrix} du_1 \\ du_2 \\ \vdots \\ du_{n-1} \end{bmatrix}, \begin{bmatrix} \beta_{1,1}(u_0) & \cdots & \beta_{1,n-1}(u_0) \\ \vdots & \ddots & \vdots \\ \beta_{n-1,1}(u_0) & \cdots & \beta_{n-1,n-1}(u_0) \end{bmatrix} \begin{bmatrix} du_1 \\ du_2 \\ \vdots \\ du_{n-1} \end{bmatrix} \right\rangle.$$

The metric tensor also has the ability to describe the local surface area of surface. Indeed, suppose that $\Omega \subseteq U$. Then the surface area of the region $\Phi[\Omega]$ on the surface is given by the integral (here $\int$ denotes an integral over a region in $\mathbb{R}^{n-1}$):

$$A = \int_\Omega \sqrt{\det\left(\begin{bmatrix} \beta_{1,1}(u_0) & \cdots & \beta_{1,n-1}(u_0) \\ \vdots & \ddots & \vdots \\ \beta_{n-1,1}(u_0) & \cdots & \beta_{n-1,n-1}(u_0) \end{bmatrix} \right)} \prod_{k=1}^{n} du_k.$$

which on a differential level takes the form:

$$dA = \sqrt{\det\left(\begin{bmatrix} \beta_{1,1}(u_0) & \cdots & \beta_{1,n-1}(u_0) \\ \vdots & \ddots & \vdots \\ \beta_{n-1,1}(u_0) & \cdots & \beta_{n-1,n-1}(u_0) \end{bmatrix} \right)} \prod_{k=1}^{n} du_k.$$

This equation can be arrived at from the Cauchy-Binet equation. However, since we won't need this general form for surface area I won't give an introduction to the above fact [check though future editions of this book]. In fact, the above is often taken to be the definition of surface area of surfaces in $\mathbb{R}^n$.

Just like we saw in the previous chapter, the Christoffel symbols play an extremely important role in the differential geometry two-dimensional surfaces sitting in $\mathbb{R}^3$. And they play a role of no less importance in the differential geometry of general surfaces sitting in $\mathbb{R}^n$, and they are defined by just a natural extension of their definitions in $\mathbb{R}^3$. On the region in the domain of a surface parametrization Φ where you've defined a Gauss map, the Christoffel symbols are the coefficients in the linear basis decomposition of the vectors $\partial^2\Phi/\partial u_k \partial u_j$ in the following basis of $\mathbb{R}^n$:

$$\left\{ \frac{\partial \Phi}{\partial u_1}, \frac{\partial \Phi}{\partial u_1}, \dots, \frac{\partial \Phi}{\partial u_{n-1}}, N \right\}$$

in front of the partials of Φ. The fact that $\left\{\frac{\partial\Phi}{\partial u_1}, \frac{\partial\Phi}{\partial u_1}, \dots, \frac{\partial\Phi}{\partial u_{n-1}}, N\right\}$ is a basis of $\mathbb{R}^n$ comes from the fact that $\left\{\frac{\partial\Phi}{\partial u_1}, \frac{\partial\Phi}{\partial u_1}, \dots, \frac{\partial\Phi}{\partial u_{n-1}}\right\}$ is a linearly independent list (since the rank of the matrix $D\Phi$ is maximal – see the third condition of a surface parametrization in Definition 6.2.2) and that N is normal to every vector in this list.

Definition 6.4.3: *Let S be a smooth $(n-1)$-dimensional surface sitting in $\mathbb{R}^n$ and $\Phi : U \to V \cap S$ a surface parametrization of S where $U \subseteq \mathbb{R}^{n-1}$ and $V \subseteq \mathbb{R}^n$ are open sets. Suppose also that $N : W \to \mathbb{R}^n$ is a Gauss map where $W \subseteq U$ is some subset of U. Then the Christoffel symbols $\Gamma_{kj}^m : W \to \mathbb{R}$ for $k, j, m \in \{1,2,\dots,n-1\}$ are the following coefficients in the linear basis decomposition of the vectors $\partial^2\Phi/\partial u_k \partial u_j$ in the basis $\left\{\frac{\partial\Phi}{\partial u_1}, \frac{\partial\Phi}{\partial u_1}, \dots, \frac{\partial\Phi}{\partial u_{n-1}}, N\right\}$:*

$$\frac{\partial^2\Phi}{\partial u_k \partial u_j} = \sum_{d=1}^{n-1}\left(\Gamma_{kj}^d \frac{\partial\Phi}{\partial u_d}\right) - \psi_{kj}N$$

(here the partials of Φ, the Christoffel symbols, and N are being evaluated at $(u_1, u_2, \dots u_{n-1}) \in W$).

The reason that the $-\psi_{kj}$'s are the coefficients in front of the N in the above equation is given by the equations for the second fundamental form stated at the end of Definition 6.4.1.

Section 5: The Minimizing Curve Theorem for Surface in $\mathbb{R}^n$

Here we finally get to the exciting task of proving our first variational theorem for general surfaces sitting in $\mathbb{R}^n$: The Minimizing Curve Theorem for general surfaces. Just like we had on two-dimensional surface sitting in $\mathbb{R}^3$, if you take two points on a general surface sitting in $\mathbb{R}^n$ and take a path between them of shortest arclength, then that path is called an **arclength minimizing curve**, or just simply **minimizing curve**.

Definition 6.5.1: *Suppose that S is a smooth $(n-1)$-dimensional surface sitting in $\mathbb{R}^n$ and let p_1 and p_2 be any two distinct points on S. Then a curve $\gamma \in C^2[t_0, t_1]$ in $\mathbb{R}^n$ that lies on the surface is called a* ***minimizing curve*** *between p_1 and p_2 if it connects the two points:*

$$\gamma(t_0) = p_1 \quad \textit{and} \quad \gamma(t_1) = p_2,$$

its derivative is nonvanishing, and for any other surface curve $\eta \in C^2[t_0, t_1]$ that connects these two points and whose derivative never vanishes, the arclength of $\gamma(t)$ is less than or equal to the arclength of $\eta(t)$:

$$L[\gamma(t)] \leq L[\eta(t)].$$

In other words, a minimizing curve between two points on a surface is a surface path of shortest arclength. The reason why we require that the derivatives of the curves involved don't vanish is always to prevent the image of the curves from having awkward corners.

Just like in the case of two-dimensional surface sitting in $\mathbb{R}^3$, we can find a necessary condition for minimizing curves on general surface by applying the Euler-Lagrange differential equation to the arclength functional, which in turn gives us a powerful way to find such minimizing curves.

We will do this exactly how we did in the previous chapter except that here we will use the hypersurface Euler-Lagrange vector differential equation since we are working on general surfaces. As for Theorem 5.4.2, the main idea in the following proof came from something that Gelfand and Fomin presented in their calculus of variations book.

Theorem 6.5.2 (n-Dimensional Version Minimizing Curve Theorem): *Suppose that S is a smooth $(n-1)$-dimensional surface sitting in $\mathbb{R}^n$ and let p_1 and p_2 be any two distinct points on S. Suppose that $\gamma \in C^2[a,b]$ is a minimizing curve between p_1 and p_2. Then for any time $t \in (a,b)$, our minimizing curve satisfies the* ***minimizing curve equation****:*

$$\gamma''(t) \in span\{\gamma'(t), N\}$$

where N is any nonzero normal vector to the surface at the point $\gamma(t)$.

Proof: The N above does not have to be the Gauss map (which is a pretty strong construction) and at each point $\gamma(t)$ N is just a nonzero normal vector to the surface at that point. The proof of this theorem is exactly the same as the proof of Theorem 5.4.2 except that here we have to do things in more variables. In this proof, let x denote the vector:

$$x = (x_1, x_2, \dots, x_n).$$

To prove this theorem, we will use the hypersurface Euler-Lagrange vector differential equation that we derived in Chapter 3 (Theorem 3.3.10). Let us take any time $t_0 \in (a,b)$ and show that the minimizing curve satisfies the minimizing curve equation at time $t = t_0$ (in other words, our strategy here is to focus on one time in the interval (a,b) at a time). Since S is a smooth surface, by Theorem 6.3.2 there exists an open set V that contains the point $\gamma(t_0)$ on the surface such that $V \cap S$ is the level set of a C^∞ function $g(x)$ whose gradient never vanishes on $V \cap S$. Since V is open, by the continuity of $\gamma(t)$ we know that there exists some small time interval $[\alpha, \beta] \subseteq (a,b)$ centered at t_0 such that $\gamma(t)$ is always inside of V, and in particular inside of $V \cap S$, for all times $t \in [\alpha, \beta]$.

Basically what we've done here is locally to $\gamma(t_0)$ we represented the surface as the level set of a C^∞ function. Why did we do this? We did this because we're about to apply the hypersurface Euler-Lagrange vector differential equation to $\gamma(t)$ at time $t = t_0$. Let us form the arclength functional J:

$$J[u_1(t), u_2(t), \dots, u_n(t)] = \int_\alpha^\beta \sqrt{\left(u_1'(t)\right)^2 + \left(u_2'(t)\right)^2 + \cdots + \left(u_n'(t)\right)^2}\, dt$$

where the domain of the functional is the set of C^2 curves $\left(u_1(t), u_2(t), \dots, u_n(t)\right)$ that lie on the surface S and that satisfy the boundary conditions of agreeing with the minimizing curve at times $t = \alpha$ and $t = \beta$:

$$\big(u_1(\alpha), u_2(\alpha), \dots, u_n(\alpha)\big) = \gamma(\alpha) \quad and \quad \big(u_1(\beta), u_2(\beta), \dots, u_n(\beta)\big) = \gamma(\beta),$$

$$\big(u_1'(\alpha), u_2'(\alpha), \dots, u_n'(\alpha)\big) = \gamma'(\alpha) \quad and \quad \big(u_1'(\beta), u_2'(\beta), \dots, u_n'(\beta)\big) = \gamma'(\beta),$$

$$\big(u_1''(\alpha), u_2''(\alpha), \dots, u_n''(\alpha)\big) = \gamma''(\alpha) \quad and \quad \big(u_1''(\beta), u_2''(\beta), \dots, u_n''(\beta)\big) = \gamma''(\beta).$$

We need all possible derivatives (0 through 2) of the curves in the domain of J to agree with the minimizing curve to make the following claim: the restriction of the minimizing curve $\gamma(t)$ to the time interval $t \in [\alpha, \beta]$ is a local minimum of the functional J. The reason why this is true should be clear. If it were not true that the restriction of $\gamma(t)$ to $t \in [\alpha, \beta]$ is a local minimum of J, then arbitrarily close to this restriction would exist a curve $\big(\widetilde{u_1}(t), \widetilde{u_2}(t), .., \widetilde{u_n}(t)\big)$ in the domain of J that has a shorter arclength than $\gamma(t)$ between the points $\gamma(\alpha)$ and $\gamma(\beta)$. But then we could construct the curve:

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) \quad if \quad t \notin [\alpha, \beta] \\ \big(\widetilde{u_1}(t), \widetilde{u_2}(t), .., \widetilde{u_n}(t)\big) \quad if \quad t \in [\alpha, \beta] \end{cases}$$

and notice that this new curve is still C^2 (because of the above boundary conditions imposed on the curves in the domain of J, like $\big(\widetilde{u_1}(t), \widetilde{u_2}(t), .., \widetilde{u_n}(t)\big)$) and that it has a shorter arclength than $\gamma(t)$ between the points p_1 and p_2. But this contradicts the fact that $\gamma(t)$ is a minimizing curve between p_1 and p_2. So the restriction of $\gamma(t)$ to $t \in [\alpha, \beta]$ indeed must be a local minimum of J.

Great! The fact that the restriction of $\gamma(t)$ to $t \in [\alpha, \beta]$ is a local minimum of J means that it must be a solution to the surface Euler-Lagrange vector differential equation of this arclength functional. Let $\left(u_{1_0}(t), u_{2_0}(t), \dots, u_{n_0}(t)\right)$ denote the restriction of the minimizing curve $\gamma(t)$ onto the interval $t \in [\alpha, \beta]$. Let F, as in our usual notation, denote the integrand of the integral in the definition of J. Since $\left(u_{1_0}(t), u_{2_0}(t), \dots, u_{n_0}(t)\right)$ is a local minimum of J, according to Theorem 3.3.1 it must satisfy the differential equation:[41]

$$\nabla_\delta J\big[u_{1_0}(t), u_{2_0}(t), \dots, u_{n_0}(t)\big] = \lambda(t) \nabla g\left(u_{1_0}(t), u_{2_0}(t), \dots, u_{n_0}(t)\right)$$

on $t \in (\alpha, \beta)$ for some real valued function $\lambda : (\alpha, \beta) \to \mathbb{R}$. Let's calculate the left-hand side of the above equation:

$$\nabla_\delta J\big[u_{1_0}(t), u_{2_0}(t), \dots, u_{n_0}(t)\big] = \left(\frac{\partial F}{\partial u_1} - \frac{d}{dt}\left(\frac{\partial F}{\partial u_1'}\right), \frac{\partial F}{\partial u_2} - \frac{d}{dt}\left(\frac{\partial F}{\partial u_2'}\right), \dots, \frac{\partial F}{\partial u_n} - \frac{d}{dt}\left(\frac{\partial F}{\partial u_n'}\right)\right).$$

Since the integrand F does not have u_k's in it, we have that $\frac{\partial F}{\partial u_k}$, are all zero for $k \in \{1,2, \dots, n\}$ and so:

[41] A comment of a similar sort to the one made in footnote 31 on page 181 applies here.

$$\nabla_\delta J\left[u_{1_0}(t), u_{2_0}(t), \dots, u_{n_0}(t)\right] = -\left(\frac{d}{dt}\left(\frac{\partial F}{\partial u_1'}\right), \frac{d}{dt}\left(\frac{\partial F}{\partial u_2'}\right), \dots, \frac{d}{dt}\left(\frac{\partial F}{\partial u_n'}\right)\right)$$

$$= -\frac{d}{dt}\left(\frac{\partial F}{\partial u_1'}, \frac{\partial F}{\partial u_2'}, \dots, \frac{\partial F}{\partial u_n'}\right)$$

$$= -\frac{d}{dt}\left(\frac{u_{1_0}'}{\sqrt{u_{1_0}'^2 + u_{2_0}'^2 + \cdots + u_{n_0}'^2}}, \frac{u_{2_0}'}{\sqrt{u_{1_0}'^2 + u_{2_0}'^2 + \cdots + u_{n_0}'^2}}, \dots, \frac{u_{n_0}'}{\sqrt{u_{1_0}'^2 + u_{2_0}'^2 + \cdots + u_{n_0}'^2}}\right)$$

$$= -\frac{d}{dt}\left(\frac{\left(u_{1_0}', u_{2_0}', \dots, u_{n_0}'\right)}{\sqrt{u_{1_0}'^2 + u_{2_0}'^2 + \cdots + u_{n_0}'^2}}\right)$$

$$= -\frac{\left(u_{1_0}'', u_{2_0}'', \dots, u_{n_0}''\right)}{\sqrt{u_{1_0}'^2 + u_{2_0}'^2 + \cdots + u_{n_0}'^2}} - \frac{d}{dt}\left(\frac{1}{\sqrt{u_{1_0}'^2 + u_{2_0}'^2 + \cdots + u_{n_0}'^2}}\right)\left(u_{1_0}', u_{2_0}', \dots, u_{n_0}'\right).$$

Notice that $\sqrt{u_{1_0}'^2 + u_{2_0}'^2 + \cdots + u_{n_0}'^2}$ is never zero since $\left(u_{1_0}(t), u_{2_0}(t), \dots, u_{n_0}(t)\right)$ is the restriction of the minimizing curve $\gamma(t)$ to $[\alpha, \beta]$ and by definition the derivative of minimizing curves never vanish (see Definition 6.5.1). So we get that out surface Euler-Lagrange vector differential equation can be rewritten a:

$$= -\frac{\left(u_{1_0}'', u_{2_0}'', \dots, u_{n_0}''\right)}{\sqrt{u_{1_0}'^2 + u_{2_0}'^2 + \cdots + u_{n_0}'^2}} - \frac{d}{dt}\left(\frac{1}{\sqrt{u_{1_0}'^2 + u_{2_0}'^2 + \cdots + u_{n_0}'^2}}\right)\left(u_{1_0}', u_{2_0}', \dots, u_{n_0}'\right)$$

$$= \lambda(t)\nabla g\big(u(t), v(t), w(t)\big).$$

Since $\left(u_{1_0}(t), u_{2_0}(t), \dots, u_{n_0}(t)\right)$ is the restriction of $\gamma(t)$ to $[\alpha, \beta]$, over $t \in (\alpha, \beta)$ the above equation can be rewritten as:

$$-\frac{\gamma''(t)}{\|\gamma'(t)\|} - \frac{d}{dt}(\|\gamma'(t)\|^{-1})\gamma'(t) = \lambda(t)\nabla g\big(\gamma(t)\big).$$

Solving for $\gamma''(t)$ finally gives that:

$$\gamma''(t) = -\|\gamma'(t)\|\lambda(t)\nabla g\big(\gamma(t)\big) - \|\gamma'(t)\|\frac{d}{dt}(\|\gamma'(t)\|^{-1})\gamma'(t).$$

In particular, this equation holds at time $t = t_0$:

$$\gamma''(t_0) = -\|\gamma'(t_0)\|\lambda(t_0)\nabla g\big(\gamma(t_0)\big) - \|\gamma'(t_0)\| \frac{d}{dt}\left(\|\gamma'(t)\|^{-1}\right)\Big|_{t=t_0} \gamma'(t_0).$$

So $\gamma''(t_0)$ is in the span of $\gamma'(t_0)$ and $\nabla g\big(\gamma(t_0)\big)$! Now, let N be any nonzero vector that is normal to the surface at the point $\gamma(t_0)$. Since both N and $\nabla g\big(\gamma(t_0)\big)$ are perpendicular to the linear subspace $T_{\gamma(t_0)}(S)$ which has dimension $(n-1)$ (that's on less than the dimension of the space we're working in: $\mathbb{R}^n$), we get that N and $\nabla g\big(\gamma(t_0)\big)$ are linear dependent and thus the above equation finally tells us that $\gamma(t)$ satisfies the minimizing curve equation at time $t = t_0$:

$$\gamma''(t_0) \in span\{\gamma'(t_0), N\}.$$

Since t_0 was chosen arbitrarily in the interval (a, b), we get that the above equation holds for all $t \in (a, b)$ and thus we have proved the theorem.

■

Note 6.5.3: As it was for minimizing curves on two dimensional surfaces sitting in $\mathbb{R}^3$, an interesting special case of the minimizing curve equation is when we parametrize our curve to be unit speed. By a similar argument as in the discussion in Note 5.4.3, we can always reparametrize any minimizing curve to be unit speed and furthermore it satisfies the property that its second derivative is constantly perpendicular to its first derivative. In other words, if $\gamma(t)$ is a unit speed parametrization of a minimizing curve, then:

$$\gamma''(t) \cdot \gamma'(t) = 0.$$

This is a powerful property since then the minimizing curve equation:

$$\gamma''(t) \in span\{\gamma'(t), N\}$$

implies that $\gamma''(t)$ is constantly linearly dependent with N – the non-zero normal vectors to the surface. In other words, we have that $\gamma''(t)$ is constantly orthogonal to the tangent plane $T_{\gamma(t)}(S)$. In a surface parametrization Φ this behavior can be written down in the following way.

Let $\big(u_1(t), u_2(t), \dots, u_{n-1}(t)\big) = \Phi^{-1}\big(\gamma(t)\big)$ be the inverse image of the unit speed minimizing curve $\gamma(t)$ under Φ (which is C^∞ by Lemma 6.2.4) for the times that $\gamma(t)$ is in the image of Φ. Then the second derivative of $\gamma(t)$ can be written as (in the following I omit the arguments of the partials of Φ; they are being evaluated at $\big(u_1(t), u_2(t), \dots, u_{n-1}(t)\big)$):

$$\gamma''(t) = \frac{d^2}{dt^2}\Big(\Phi\big(u_1(t), u_2(t), \dots, u_{n-1}(t)\big)\Big) = \frac{d}{dt}\left(\sum_{k=1}^{n-1} \frac{\partial \Phi}{\partial u_k} u_k'(t)\right)$$

$$= \sum_{k=1}^{n-1}\left(\frac{\partial \Phi}{\partial u_k} u_k''(t)\right) + \sum_{k=1}^{n-1}\sum_{j=1}^{n-1}\left(\frac{\partial^2 \Phi}{\partial u_k \partial u_j} u_k'(t) u_j'(t)\right)$$

$$= \sum_{k=1}^{n-1} \left(\frac{\partial \Phi}{\partial u_k} u_k''(t) \right) + \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} \left(\left(\sum_{m=1}^{n-1} \left(\Gamma_{k,j}^{m} \frac{\partial \Phi}{\partial u_m} \right) - \psi_{k,j} N \right) u_k'(t) u_j'(t) \right).$$

Rearranging the terms in the sums gives that:

$$\gamma''(t) = \sum_{m=1}^{n-1} \left(\left(u_m''(t) + \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} \left(\Gamma_{k,j}^{m} u_k'(t) u_j'(t) \right) \right) \frac{\partial \Phi}{\partial u_m} \right) - \left(\sum_{k}^{n-1} \sum_{j=1}^{n-1} \psi_{k,j} u_k'(t) u_j'(t) \right) N.$$

This is the linear basis decomposition of $\gamma''(t)$ in the basis $\left\{ \frac{\partial \Phi}{\partial u_1}, \frac{\partial \Phi}{\partial u_2}, \dots, \frac{\partial \Phi}{\partial u_{n-1}}, N \right\}$ of $\mathbb{R}^n$. Since $\gamma''(t)$ is linearly dependent with N, or equivalently is orthogonal to $T_{\gamma(t)}(S)$ and thus perpendicular to each $\frac{\partial \Phi}{\partial u_m}$, the above equation then implies that $\gamma''(t)$ satisfies the system of differential equations:

$$\forall m \in \{1,2,\dots,n-1\}, \qquad u_m''(t) + \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} \left(\Gamma_{k,j}^{m} u_k'(t) u_j'(t) \right) = 0$$

since the coefficients in front of the $\partial \Phi / \partial u_m$'s in the previous equation must all be equal to zero. This system of differential equations for the unit speed minimizing curves are extremely important because they describe the minimizing curves entirely in terms of the entries of the metric tensor (since the Christoffel symbols can be solved for in terms of the entries of the metric tensor). As an example of an application of this fact, the above form of the minimizing curve equation for example shows that minimizing curves are conserved under isometries (metric preserving maps).

Section 6: Rotation in the Domain of a Functional Trick

As we have seen multiple times, the field of variational differential geometry can sometimes involve long and scary computations. For such a reason, tricks and techniques that help shorten such calculations and achieve the same thing are often sought after and needed. One such technique is the rotation of curves or surfaces in a functional's domain in order to compute the variational derivative at curve evaluated at a single point. Let me demonstrate this technique on a functional whose domain is a set of curves. This trick applied to other types of functionals, including functionals over spaces of multivariable functions, works exactly the same.

Let's suppose that J is a functional of the form:

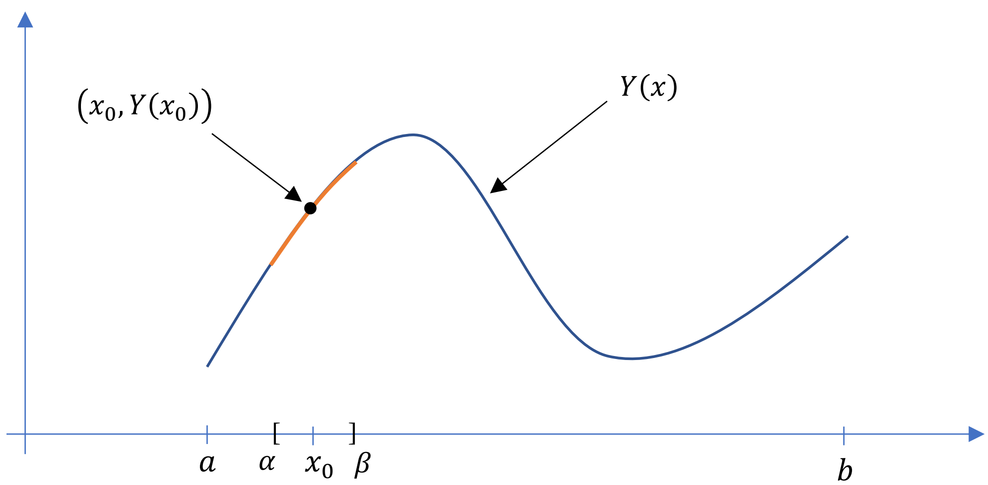
$$J[y] = \int_a^b F(x, y, y') dx$$

where $F \in C^2[\mathbb{R}^3]$ and J's domain is the set of curves $y(x) \in C^2[a,b]$ that satisfy the boundary conditions:

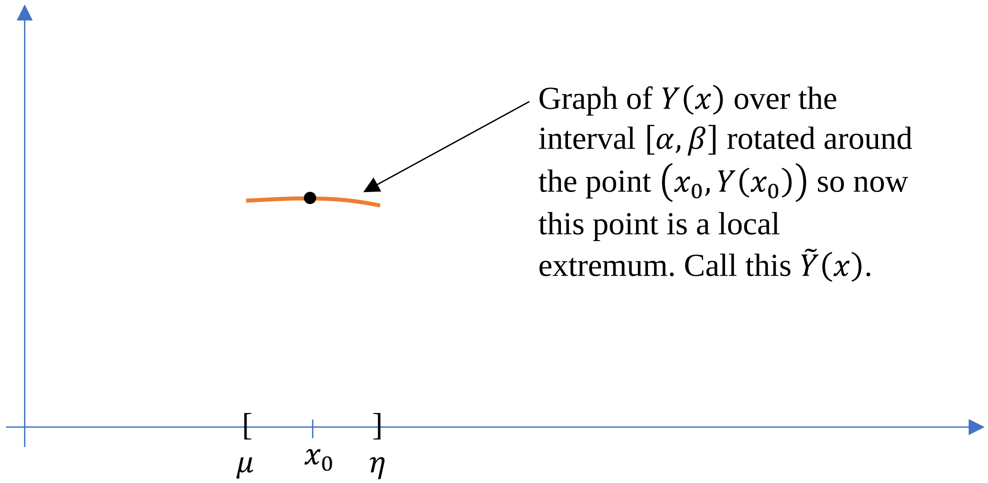
$$y(a) = A \quad and \quad y(b) = B$$

where A and B are two real numbers. Suppose also that J is a functional that measures some geometric property of curves – properties that are invariant under translations and rotations (rotation invariance is what we will need). An example of such a property is arclength. Now, suppose that we have a curve $Y(x)$ in the domain of J at which we want to compute the functional derivative of J at: $\frac{\delta J}{\delta y}[Y]$.

Let's also suppose that in this situation directly plugging $Y(x)$ into the equation for the functional derivative (which is the left-hand side of the Euler-Lagrange differential equation) would result in a massive calculation and so we desperately need a trick that helps circumvent such a long a calculation and that achieves the same result. This rotation technique that I am about to show you goes about calculating the functional derivative at each single point $x_0 \in (a,b)$ at a time. The values of the functional derivative at the boundary boundaries will then follow from the continuity of the functional derivative as a function of x. So fix any point $x_0 \in (a,b)$ where we want to evaluate the functional derivative of J at $Y(x)$.



Now, there exists a small interval $[\alpha,\beta] \subseteq (a,b)$ centered at x_0 such that if you rotate the graph of $Y(x)$ over this interval $[\alpha,\beta]$ (orange in the above picture) around the point $(x_0, Y(x_0))$ so as to make the tangent line at the point $(x_0, Y(x_0))$ flat (forget about the rest of the blue curve for now):



it will still be the graph of a function (passes the vertical line test for the younger generation). Let θ denote the radians that you rotated that piece of the curve by. Let's also suppose that you rotated it so that $|\theta| \leq \frac{\pi}{2}$ (you can always choose such a direction of rotation). Let's call this new curve that we get $\tilde{Y}(x)$ whose domain is some interval $[\mu, \eta]$. Now let us form a functional $\tilde{J}$ whose domain is the set of curves $\tilde{y}(x) \in C^2[\mu, \eta]$ that satisfy the boundary conditions of agreeing with $\tilde{Y}(x)$ up to the second derivative at $x = \delta$ and $x = \eta$:

$$\tilde{y}(\mu) = \tilde{Y}(\mu) \quad and \quad \tilde{y}(\eta) = \tilde{Y}(\eta),$$

$$\tilde{y}'(\mu) = \tilde{Y}'(\mu) \quad and \quad \tilde{y}'(\eta) = \tilde{Y}'(\eta),$$

$$\tilde{y}''(\mu) = \tilde{Y}''(\mu) \quad and \quad \tilde{y}''(\eta) = \tilde{Y}''(\eta),$$

and for any $\tilde{y}$ in the domain of $\tilde{J}$, $\tilde{J}[\tilde{y}]$ is the measure of that same geometric property that J measures on the curves in its domain (such as arclength). Suppose that $\tilde{J}$ can be written in the form:

$$\tilde{J}[\tilde{y}] = \int_{\mu}^{\eta} \tilde{F}(s, \tilde{y}, \tilde{y}')dx$$

for some $\tilde{F} \in C^2[\mathbb{R}^3]$. Now here comes the heart of the matter: if you show that the functional derivative of $\tilde{J}$ at $\tilde{Y}(x)$ evaluated at $x = x_0$ is equal to the functional derivative of J at $Y(x)$ evaluated at $x = x_0$:

$$\frac{\delta\tilde{J}}{\delta\tilde{y}}[\tilde{Y}]\bigg|_{x=x_0} = \frac{\delta J}{\delta y}[Y]\bigg|_{x=x_0}.$$

So basically, rotating a curve in the functional's domain and evaluating the functional derivative that way gives the same answer. Let's prove this. First of all, you might wonder why we even do this whole thing with the rotation of the curve. The answer is that the computation of the functional derivative at a curve evaluated at a point is sometimes easier when that point is a relative extremum. For example, if the expression for the functional has gradient terms in it, then all of those terms will go away when you compute its functional derivative at the local extrema.

Ok, so let's prove that the functional derivative of $\tilde{J}$ at $\tilde{Y}(x)$ evaluated at $x = x_0$ is equal to the functional derivative of J at $Y(x)$ evaluated at $x = x_0$. In this proof we will be borrowing ideas from distribution theory and the technique used in this proof is used in many places of analysis. I will explain the idea behind the following proof after I present it.

Take any nonnegative function $\varphi \in C^2[\mathbb{R}]$ that is zero on $[\mu, \eta]^c$, that satisfies the boundary conditions:

$$\varphi(\mu) = 0 \quad and \quad \varphi(\eta) = 0,$$

$$\varphi'(\mu) = 0 \quad and \quad \varphi'(\eta) = 0,$$

$$\varphi''(\mu) = 0 \quad and \quad \varphi''(\eta) = 0,$$

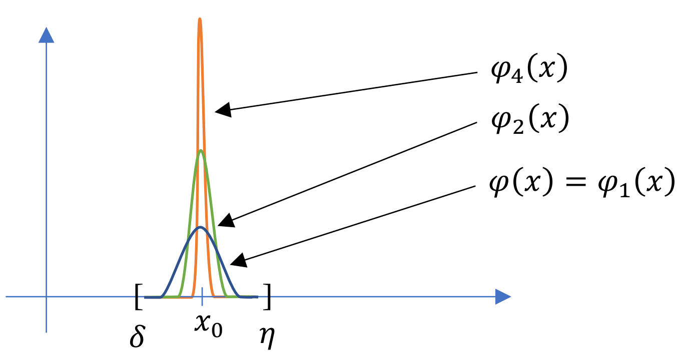
and the integral:

$$\int_{-\infty}^{\infty} \varphi(x)ds = \int_{\mu}^{\eta} \varphi(x)ds = 1.$$

It will be extremely important later on that $\varphi(x)$ is nonnegative. Such functions are not hard to construct and the equation for the spike function in the proof of Lemma 1.3.1 shows a good way to write an explicit equation for such a function. Now let us define the sequence of functions $\{\varphi_n(x)\}_{n=1}^{\infty}$ defined by: for any $n \in \mathbb{Z}_+$,

$$\varphi_n(x) = n\varphi\big(x_0 + n(x - x_0)\big).$$

If you graph this sequence of $\varphi_n(x)$'s for several n's you will see that these φ_n's will get skinnier onto the point x_0 and become really tall.



For those who know what the Dirac delta function is, the φ_n's will look more and more like the Dirac delta function $\delta(x - x_0)$. Intuitively that's what we're are going to use these φ_n's for: we will use them to get approximate Dirac delta function behavior. And these φ_n's don't converge onto x_0 in the fashion described above just randomly. They converge in such a ways that their areas under the curve are always equal to one: for any $n \in \mathbb{Z}_+$ (here I do a change of variables),

$$\int_{\mu}^{\eta} \varphi_n(x)dx = \int_{-\infty}^{\infty} \varphi_n(x)dx = \int_{-\infty}^{\infty} n\varphi\big(x_0 + n(x - x_0)\big)dx = \int_{-\infty}^{\infty} \varphi(u)du = 1.$$

Another very important feature of these $\varphi_n(x)$'s is that the size of the place on the x-axis where they are nonzero shrinks to zero. Specifically, each $\varphi_n(x)$ is zero outside of the interval (careful not to mix η with n):

$$\left[x_0 - \frac{x_0 - \mu}{n}, x_0 + \frac{\eta - x_0}{n}\right].$$

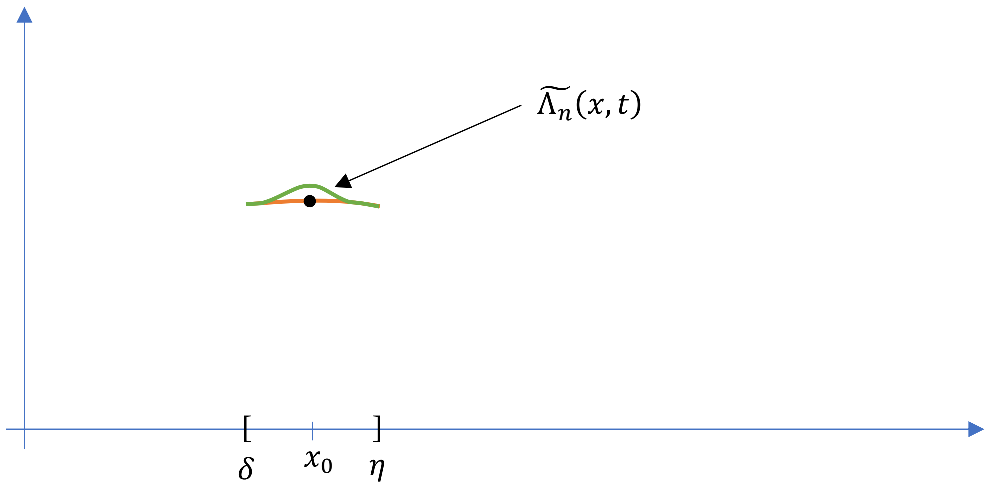
And the lengths of these intervals go to zero as $n \to \infty$. Now, let us form the sequence of 2-smooth linear flows $\left\{\widetilde{\Lambda_n} : [\mu, \eta] \times [-1, 1] \to \mathbb{R}\right\}_{n=1}^{\infty}$ defined by: for any $n \in \mathbb{Z}_+$,

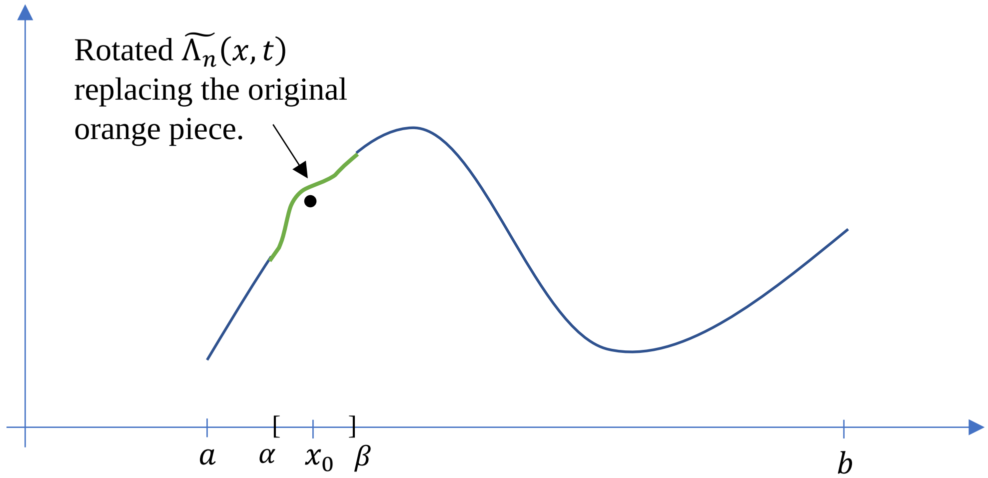
$$\widetilde{\Lambda_n}(x, t) = \tilde{Y}(x) + \varphi_n(x)t.$$

Intuitively speaking this flow flows through $\tilde{Y}(x)$ at time $t = 0$ with flows speed $\varphi_n(x)$. Notice that these flows always stays inside of the domain of $\tilde{J}$ (meaning, for each fix $t \in [-1, 1]$, $\widetilde{\Lambda_n}(x, t)$ is inside of $\mathrm{dom}(\tilde{J})$). Notice also that by considering where $\varphi_n(x)$ is equal to zero, we see that for each $n \in \mathbb{Z}_+$, the flow $\widetilde{\Lambda_n}(x, t)$ is not deforming $\tilde{Y}(x)$ outside of the interval:

$$\left[x_0 - \frac{x_0 - \mu}{n}, x_0 + \frac{\eta - x_0}{n}\right]$$

as t varies over $[-1, 1]$. And the flows always flow up since $\varphi_n(x)$'s are all nonnegative. In other words, $\frac{\partial \widetilde{\Lambda_n}}{\partial t}(x, t)$ is always nonnegative for all n, x, and t. Now, take any $n \in \mathbb{Z}_+$. For each time $t \in [-1,1]$ take the curve $\widetilde{\Lambda_n}(x, t)$ and rotate it around the point $\big(x_0, Y(x_0)\big)$ by $-\theta$ radians and replace the section of the curve of $Y(x)$ over $[\alpha, \beta]$ with the rotated $\widetilde{\Lambda_n}(x, t)$ curve.





Call the new curve $Y(x)$ with the new green section replacement over the interval $[\alpha, \beta]$ above $\Lambda_n(x,t)$ (this new curve is C^2 because $\widetilde{\Lambda_n}(x,t)$ satisfies the boundary conditions that all of its derivatives up to order 2 are equal to that of $\tilde{Y}(x)$ at the boundaries). It's not hard to see that for each $n \in \mathbb{Z}_+$ there exists a small enough time interval $[-\Delta_n, \Delta_n] \subseteq [-1, 1]$ centered at 0 such that for any time $t \in [-\Delta_n, \Delta_n]$, $\Lambda_n(x,t)$ is the graph of a function (to see this just analyze the maximum and minimum of the derivatives involved). With this we can form a new sequence of flows $\{\Lambda_n : [\alpha, \beta] \times [-\Delta_n, \Delta_n] \to \mathbb{R}\}_{n=1}^{\infty}$. Technically these are not classical *linear* flows as defined in Definition 1.2.8 since they is not of the form $y_1(x) + y_2(x)t$, but we will still call them flows because for each fixed time $t \in [-\Delta_n, \Delta_n]$, $\Lambda(x,t)$ is a function of x which is in fact inside of the domain of J. So $\{\Lambda_n : [\alpha, \beta] \times [-\Delta_n, \Delta_n] \to \mathbb{R}\}_{n=1}^{\infty}$ are flows that stay inside of the domain of J. Notice that they pass through $Y(x)$ at time $t = 0$. By geometric consideration we can also see that the flow $\Lambda_n(x,t)$ for each $n \in \mathbb{Z}_+$ is not deforming $Y(x)$ outside of the interval:

$$\left[x_0 - \frac{x_0 - \mu}{n}\cos(\theta), x_0 + \frac{\eta - x_0}{n}\cos(\theta)\right]$$

as t varies over $[-\Delta_n, \Delta_n]$ since its unrotated cousin $\widetilde{\Lambda_n}(x,t)$ is not deforming $\tilde{Y}(x)$ outside of the interval $[x_0 - (x_0 - \mu)/n, x_0 + (\eta - x_0)/n]$. This is important because the lengths of these intervals are also going to zero. Notice also that $\Lambda_n(x,t)$ always moves upwards or downwards. In other words, $\frac{\partial \Lambda_n}{\partial t}(x,t)$ is either always positive or always negative for all n, x, and t.

As the flows Λ_n's flow, the geometric quantity that J measures of the curve $Y(x)$ is not changing outside of the interval $[\alpha, \beta]$ since the Λ_n's are not deforming $Y(x)$ there. So let us restrict our functional J's attention only to the interval $[\alpha, \beta]$ and call that new restricted functional J_R:

$$J_R[y] = \int_{\alpha}^{\beta} F(x, y, y')dx.$$

It's easy to see that the variational derivative of J_r at $Y(x)$ is equal to the variational derivative of J at $Y(x)$ over the interval $x \in [\alpha, \beta]$ since the variational derivative only depends on the integrand and not the integration bounds. Now let us see how the functional and its rotated form are changing as t varies over the interval $[-\Delta_n, \Delta_n]$ for each $n \in \mathbb{Z}_+$. Since both J_R and $\tilde{J}$ measure the same geometric quantity, for any $n \in \mathbb{Z}_+$ and any time $t \in [-\Delta_n, \Delta_n]$,

$$J_R[\Lambda_n(x,t)] = \tilde{J}\left[\widetilde{\Lambda_n}(x,t)\right].$$

In particular, for any $n \in \mathbb{Z}_+$ their derivatives are equal at $t = 0$:

$$\frac{d}{dt}\left(J_R[\Lambda_n(x,t)]\right)\Big|_{t=0} = \frac{d}{dt}\left(\tilde{J}\left[\widetilde{\Lambda_n}(x,t)\right]\right)\Big|_{t=0}.$$

By Lemma 3.6.1 the above equation can be rewritten as:[42]

$$\int_{\alpha}^{\beta} \frac{\delta J_R}{\delta y}[Y]\frac{\partial \Lambda_n}{\partial t}(x,0)dx = \int_{\mu}^{\eta} \frac{\delta \tilde{J}}{\delta \tilde{y}}[\tilde{Y}]\frac{\partial \widetilde{\Lambda_n}}{\partial t}(x,0)dx.$$

We are soon going to take the limit of both sides as $n \to \infty$. But first, since $\widetilde{\Lambda_n}$ and Λ_n deform the curves $\tilde{Y}(x)$ and $Y(x)$ only on the intervals $[x_0 - (x_0 - \mu)/n, x_0 + (\eta - x_0)/n]$ and $\left[x_0 - \left((x_0 - \mu)/n\right)\cos(\theta), x_0 + \left((\eta - x_0)/n\right)\cos(\theta)\right]$ respectively, $\frac{\partial \widetilde{\Lambda_n}}{\partial t}$ and $\frac{\partial \Lambda_n}{\partial t}$ are zero outside of the intervals $[x_0 - (x_0 - \mu)/n, x_0 + (\eta - x_0)/n]$ and $\left[x_0 - \left((x_0 - \mu)/n\right)\cos(\theta), x_0 + \left((\eta - x_0)/n\right)\cos(\theta)\right]$ respectively. So the above integral equation can be rewritten as:

$$\int_{x_0-\frac{x_0-\mu}{n}\cos(\theta)}^{x_0+\frac{\eta-x_0}{n}\cos(\theta)} \frac{\delta J_R}{\delta y}[Y]\frac{\partial \Lambda_n}{\partial t}(x,0)dx = \int_{x_0-\frac{x_0-\mu}{n}}^{x_0+\frac{\eta-x_0}{n}} \frac{\delta \tilde{J}}{\delta \tilde{y}}[\tilde{Y}]\frac{\partial \widetilde{\Lambda_n}}{\partial t}(x,0)dx.$$

By the Mean Value Theorem for Integrals, there exists numbers:

$$x_1 \in \left[x_0 - \frac{x_0 - \mu}{n}, x_0 + \frac{\eta - x_0}{n}\right],$$

$$\widetilde{x_1} \in \left[x_0 - \frac{x_0 - \mu}{n}\cos(\theta), x_0 + \frac{\eta - x_0}{n}\cos(\theta)\right],$$

such that:

$$\int_{x_0-\frac{x_0-\mu}{n}\cos(\theta)}^{x_0+\frac{\eta-x_0}{n}\cos(\theta)} \frac{\delta J_R}{\delta y}[Y]\frac{\partial \Lambda_n}{\partial t}(x,0)dx = \frac{\delta J_R}{\delta y}[Y]\Big|_{x=x_1}\int_{x_0-\frac{x_0-\mu}{n}\cos(\theta)}^{x_0+\frac{\eta-x_0}{n}\cos(\theta)} \frac{\partial \Lambda_n}{\partial t}(x,0)dx,$$

[42] Technically the way Lemma 3.6.1 was stated it only applies to linear flows. However nowhere in the proof did we use the fact that the flow was linear and so the result holds for more general flows as well like Λ here.

$$\int_{x_0-\frac{x_0-\mu}{n}}^{x_0+\frac{\eta-x_0}{n}} \frac{\delta\tilde{J}}{\delta\tilde{y}}[\tilde{Y}]\frac{\partial\widetilde{\Lambda_n}}{\partial t}(x,0)dx = \frac{\delta\tilde{J}}{\delta\tilde{y}}[\tilde{Y}]\bigg|_{x=\widetilde{x_1}} \int_{x_0-\frac{x_0-\mu}{n}}^{x_0+\frac{\eta-x_0}{n}} \frac{\partial\widetilde{\Lambda_n}}{\partial t}(x,0)dx.$$

Here I used the facts that $\frac{\partial\Lambda_n}{\partial t}$ and $\frac{\partial\widetilde{\Lambda_n}}{\partial t}$ are always either nonnegative or nonpositive. This turns the previous integral equation into:

$Equation$ 6.5.1:

$$\frac{\delta J_R}{\delta y}[Y]\bigg|_{x=x_1} \int_{x_0-\frac{x_0-\mu}{n}\cos(\theta)}^{x_0+\frac{\eta-x_0}{n}\cos(\theta)} \frac{\partial\Lambda_n}{\partial t}(x,0)dx = \frac{\delta\tilde{J}}{\delta\tilde{y}}[\tilde{Y}]\bigg|_{x=\widetilde{x_1}} \int_{x_0-\frac{x_0-\mu}{n}}^{x_0+\frac{\eta-x_0}{n}} \frac{\partial\widetilde{\Lambda_n}}{\partial t}(x,0)dx.$$

Now what are the above integrals equal to? The integral on the right-hand side is equal to:

$$\int_{x_0-\frac{x_0-\mu}{n}}^{x_0+\frac{\eta-x_0}{n}} \frac{\partial\widetilde{\Lambda_n}}{\partial t}(x,0)dt = \int_{x_0-\frac{x_0-\mu}{n}}^{x_0+\frac{\eta-x_0}{n}} \varphi_n(x)dt = 1.$$

What about the integral on the left-hand side of the previous equation? Well, let's write:

$$\int_{x_0-\frac{x_0-\mu}{n}\cos(\theta)}^{x_0+\frac{\eta-x_0}{n}\cos(\theta)} \frac{\partial\Lambda_n}{\partial t}(x,0)dx = \frac{d}{dt}\left(\int_{x_0-\frac{x_0-\mu}{n}\cos(\theta)}^{x_0+\frac{\eta-x_0}{n}\cos(\theta)} \Lambda_n(x,t)dx\right)\Bigg|_{t=0}$$

$$= \lim_{\varepsilon\to 0}\left(\frac{\int_{x_0-\frac{x_0-\mu}{n}\cos(\theta)}^{x_0+\frac{\eta-x_0}{n}\cos(\theta)} \Lambda_n(x,\varepsilon)dx - \int_{x_0-\frac{x_0-\mu}{n}\cos(\theta)}^{x_0+\frac{\eta-x_0}{n}\cos(\theta)} \Lambda_n(x,0)dx}{\varepsilon}\right)$$

$$= \lim_{\varepsilon\to 0}\left(\frac{\int_{x_0-\frac{x_0-\mu}{n}\cos(\theta)}^{x_0+\frac{\eta-x_0}{n}\cos(\theta)} \big(\Lambda_n(x,\varepsilon) - \Lambda_n(x,0)\big)dx}{\varepsilon}\right).$$

Since area is invariant under rotations, the area between $\Lambda_n(x,t)$ and $Y(x) = \Lambda_n(x,0)$ is equal to that of the area between $\widetilde{\Lambda_n}(x,t)$ and $\tilde{Y}(x) = \widetilde{\Lambda_n}(x,0)$ (even up to sign since we rotated less than $\frac{\pi}{2}$ radians). So, the above limit is equal to:

$$\lim_{\varepsilon\to 0}\left(\frac{\int_{x_0-\frac{x_0-\mu}{n}}^{x_0+\frac{\eta-x_0}{n}}\left(\widetilde{\Lambda_n}(x,\varepsilon)-\widetilde{\Lambda_n}(x,0)\right)dx}{\varepsilon}\right)$$

$$=\lim_{\varepsilon\to 0}\left(\frac{\int_{x_0-\frac{x_0-\mu}{n}}^{x_0+\frac{\eta-x_0}{n}}\widetilde{\Lambda_n}(x,\varepsilon)dx-\int_{x_0-\frac{x_0-\mu}{n}}^{x_0+\frac{\eta-x_0}{n}}\widetilde{\Lambda_n}(x,0)dx}{\varepsilon}\right)$$

$$=\frac{d}{dt}\left(\int_{x_0-\frac{x_0-\mu}{n}}^{x_0+\frac{\eta-x_0}{n}}\widetilde{\Lambda_n}(x,t)dx\right)\Bigg|_{t=0}=\int_{x_0-\frac{x_0-\mu}{n}}^{x_0+\frac{\eta-x_0}{n}}\frac{\partial\widetilde{\Lambda_n}}{\partial t}(x,0)dx.$$

And this last integral we already showed is equal to 1. So, we gave that for any $n\in\mathbb{Z}_+$,

$$\int_{x_0-\frac{x_0-\mu}{n}}^{x_0+\frac{\eta-x_0}{n}}\frac{\partial\widetilde{\Lambda_n}}{\partial t}(x,0)dt=1.$$

With this we finally have that Equation 6.5.1 can be rewritten as:

$$\frac{\delta J_R}{\delta y}[Y]\bigg|_{x=x_1}=\frac{\delta\tilde{J}}{\delta\tilde{y}}[\tilde{Y}]\bigg|_{x=\widetilde{x_1}}.$$

Taking the limit of both sides as $n\to\infty$ will make both $x_1,\widetilde{x_1}\to x_0$ and thus by the continuity of the functional derivatives as functions of x we get that:

$$\frac{\delta J_R}{\delta y}[Y]\bigg|_{x=x_0}=\frac{\delta\tilde{J}}{\delta\tilde{y}}[\tilde{Y}]\bigg|_{x=x_0}.$$

Or since $\frac{\delta J_R}{\delta y}=\frac{\delta J}{\delta y}$,

$$\frac{\delta J}{\delta y}[Y]\bigg|_{x=x_0}=\frac{\delta\tilde{J}}{\delta\tilde{y}}[\tilde{Y}]\bigg|_{x=x_0}.$$

This proves the claim.

I do want to explain the idea behind the above proof. Fundamentally we used the fact that the functional is invariant under a rotation of its domain because it measures some geometric property of the elements in its domain (curves that is). Thus variational changes in the functional are invariant under such rotations. Since differential changes in the functional can be expressed in the integral form (I omit integration bounds):

$$dJ = \int \frac{\delta J}{\delta y} d\Lambda dx$$

(Lemma 3.6.1), we get that such differential integral quantities are also invariant under rotations. Then we proceed to use a trick that is used throughout many branches of analysis: if you want to evaluate an integrand integrated against a variable function, then use variable functions that have very spiky positive behavior (or: very "Dirac delta function" behavior) around a point to find the value of that integrand at that point. Here the integrand that we are trying to evaluate is $\frac{\delta J}{\delta y}$ and our variable functions on a differential level are $d\Lambda$. In the language of distribution theory, we are trying to find the values of the function that generates the distribution:

$$u[\phi] = \int \frac{\delta J}{\delta y} \phi(x) dx.$$

Our proof above is just a rigorous version of these arguments.

This trick works in higher dimensions as well when we deal with surfaces in $\mathbb{R}^3$ or in more general $\mathbb{R}^n$. In those cases, in order to evaluate the functional derivative at each point we rotate surface so that that point becomes a local extremum and then calculate the functional derivative at that point on the rotated surface. To illustrate the power of this technique, let me show you how it can be used to simplify the calculation done in Theorem 5.6.2. In that proof we had to show that the functional derivative of the "local total Gaussian curvature" functional:

$$J[h] = \iint_{\Omega} \frac{h_{xx}h_{yy} - h_{xy}^2}{\sqrt{1 + h_x^2 + h_y^2}^3} dxdy$$

at any surface h is equal to zero (here we are using the word "surface" to colloquially mean a real-valued function of two variables). Notice that this functional does measure a geometric property of the surfaces h's in its domain: it measures their total Gaussian curvature, or more explicitly the surface integral of the Gaussian curvature over them, which is a geometric property since surface integrals and Gaussian curvature are geometric quantities. So let us apply the higher dimensional version of the trick discussed above here to show that the functional derivative of J is indeed zero at any of the surfaces in its domain. Pick any surface h_0 in the domain of the above functional J (I decided to call it h_0 here in order to differentiate between the surface where we are trying to compute the functional derivative at and the variable of J). Take any point $(x_0, y_0) \in \Omega$ and rotate the surface around the point $\big(x_0, y_0, h_0(x_0, y_0)\big)$ in a similar fashion as described above with curves such that (x_0, y_0) becomes a local extremum of the rotated piece $\widetilde{h_0}$. The rotated version of the functional as described in the above trick here will be of the form:

$$\tilde{J}[\tilde{h}] = \iint_{\tilde{\Omega}} \frac{\tilde{h}_{xx}\tilde{h}_{yy} - \tilde{h}_{xy}^2}{\sqrt{1 + \tilde{h}_x^2 + \tilde{h}_y^2}^3} dxdy$$

where the domain of this functional is the set of $\tilde{h} \in C^2\left[\tilde{\Omega}\right]$ that satisfy the property that all of their partial derivatives up to order 2 (even of order 0) match with those of $\widetilde{h_0}$ on the boundary $\partial\tilde{\Omega}$. The reason that the integrand looks the same is because we are still integrating the Gaussian curvature here (we're measuring the total Gaussian curvature of $\tilde{h}$ over $\tilde{\Omega}$). The nice thing about (x_0, y_0) being a local extremum of the rotated piece $\widetilde{h_0}$ is that:

$$\nabla\widetilde{h_0}(x_0, y_0) = \left(\widetilde{h_0}_x(x_0, y_0), \widetilde{h_0}_y(x_0, y_0)\right) = 0.$$

Let $\tilde{F}$ be the integrand in the above expression for $\tilde{J}$. We have that the functional derivative of $\tilde{J}$ at $\widetilde{h_0}$ evaluated at $(x, y) = (x_0, y_0)$ is equal to (the F's here are being evaluated at $\left(x, \widetilde{h_0}, \widetilde{h_0}_x, \widetilde{h_0}_y, \widetilde{h_0}_{xx}, \widetilde{h_0}_{xy}, \widetilde{h_0}_{yy}\right)$ and the $\widetilde{h_0}$'s are being evaluated (x_0, y_0)):

$$\frac{\delta\tilde{J}}{\delta\tilde{h}}\left[\widetilde{h_0}\right]\Bigg|_{(x,y)=(x_0,y_0)}$$

$$= \left(\frac{\partial\tilde{F}}{\partial\tilde{h}} - \frac{\partial}{\partial x}\left(\frac{\partial\tilde{F}}{\partial\tilde{h}_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\tilde{F}}{\partial\tilde{h}_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial\tilde{F}}{\partial\tilde{h}_{xx}}\right) + \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial\tilde{F}}{\partial\tilde{h}_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial\tilde{F}}{\partial\tilde{h}_{yy}}\right)\right)\Bigg|_{(x,y)=(x_0,y_0)}.$$

Let's break down the computation of each term on the right-hand side separately. In the following remember to keep in mind that $\widetilde{h_0}_x(x_0, y_0), \widetilde{h_0}_y(x_0, y_0) = 0$.

Calculating the $\frac{\partial\tilde{F}}{\partial\tilde{h}}\Big|_{(x,y)=(x_0,y_0)}$ term gives:

$$\frac{\partial\tilde{F}}{\partial\tilde{h}}\Bigg|_{(x,y)=(x_0,y_0)} = 0.$$

Calculating the $\frac{\partial}{\partial x}\left(\frac{\partial\tilde{F}}{\partial\tilde{h}_x}\right)\Big|_{(x,y)=(x_0,y_0)}$ term gives:

$$\frac{\partial}{\partial x}\left(\frac{\partial\tilde{F}}{\partial\tilde{h}_x}\right)\Bigg|_{(x,y)=(x_0,y_0)} = \frac{\partial}{\partial x}\left(-3\frac{\left(\widetilde{h_0}_{xx}\widetilde{h_0}_{yy} - \widetilde{h_0}_{xy}^2\right)\widetilde{h_0}_x}{\sqrt{1 + \widetilde{h_0}_x^2 + \widetilde{h_0}_y^2}^5}\right) = -3\frac{\left(\widetilde{h_0}_{xx}\widetilde{h_0}_{yy} - \widetilde{h_0}_{xy}^2\right)\widetilde{h_0}_{xx}}{\sqrt{1 + \widetilde{h_0}_x^2 + \widetilde{h_0}_y^2}^5}.$$

Calculating the $\frac{\partial}{\partial y}\left(\frac{\partial\tilde{F}}{\partial\tilde{h}_y}\right)\Big|_{(x,y)=(x_0,y_0)}$ term gives:

$$\frac{\partial}{\partial y}\left(\frac{\partial\tilde{F}}{\partial\tilde{h}_y}\right)\Bigg|_{(x,y)=(x_0,y_0)} = \frac{\partial}{\partial y}\left(-3\frac{\left(\widetilde{h_0}_{xx}\widetilde{h_0}_{yy} - \widetilde{h_0}_{xy}^2\right)\widetilde{h_0}_y}{\sqrt{1 + \widetilde{h_0}_x^2 + \widetilde{h_0}_y^2}^5}\right) = -3\frac{\left(\widetilde{h_0}_{xx}\widetilde{h_0}_{yy} - \widetilde{h_0}_{xy}^2\right)\widetilde{h_0}_{yy}}{\sqrt{1 + \widetilde{h_0}_x^2 + \widetilde{h_0}_y^2}^5}.$$

Calculating the $\frac{\partial^2}{\partial x^2}\left(\frac{\partial \tilde{F}}{\partial \tilde{h}_{xx}}\right)\Big|_{(x,y)=(x_0,y_0)}$ term gives:

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial \tilde{F}}{\partial \tilde{h}_{xx}}\right)\Bigg|_{(x,y)=(x_0,y_0)} = \frac{\partial^2}{\partial x^2}\left(\frac{\widetilde{h_0}_{yy}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^3}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\widetilde{h_0}_{yyx}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^3} - 3\frac{\widetilde{h_0}_{yy}\left(\widetilde{h_0}_x\widetilde{h_0}_{xx}+\widetilde{h_0}_y\widetilde{h_0}_{yx}\right)}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^5}\right)$$

$$= \frac{\widetilde{h_0}_{yyxx}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^3} - 3\frac{\widetilde{h_0}_{yy}\left(\widetilde{h_0}_{xx}^2+\widetilde{h_0}_{xy}^2\right)}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^5}.$$

Calculating the $\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial \tilde{F}}{\partial \tilde{h}_{xy}}\right)\Big|_{(x,y)=(x_0,y_0)}$ term gives:

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial \tilde{F}}{\partial \tilde{h}_{xy}}\right)\Bigg|_{(x,y)=(x_0,y_0)} = \frac{\partial^2}{\partial x \partial y}\left(-2\frac{\widetilde{h_0}_{xy}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^3}\right)$$

$$= -2\frac{\partial}{\partial y}\left(\frac{\widetilde{h_0}_{xyy}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^3} - 3\frac{\widetilde{h_0}_{xy}\left(\widetilde{h_0}_x\widetilde{h_0}_{xy}+\widetilde{h_0}_y\widetilde{h_0}_{yy}\right)}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^5}\right)$$

$$= -2\frac{\widetilde{h_0}_{xyyx}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^3} + 6\frac{\widetilde{h_0}_{xy}\left(\widetilde{h_0}_{xx}\widetilde{h_0}_{xy}+\widetilde{h_0}_{yx}\widetilde{h_0}_{yy}\right)}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^5}.$$

Calculating the $\frac{\partial^2}{\partial y^2}\left(\frac{\partial \tilde{F}}{\partial \tilde{h}_{yy}}\right)\Big|_{(x,y)=(x_0,y_0)}$ term gives:

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial \tilde{F}}{\partial \tilde{h}_{yy}}\right)\Bigg|_{(x,y)=(x_0,y_0)} = \frac{\partial^2}{\partial y^2}\left(\frac{\widetilde{h_0}_{xx}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^3}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\widetilde{h_0}_{xxy}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^3} - 3\frac{\widetilde{h_0}_{xx}\left(\widetilde{h_0}_x\widetilde{h_0}_{xy}+\widetilde{h_0}_y\widetilde{h_0}_{yy}\right)}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^5}\right)$$

$$= \frac{\widetilde{h_0}_{xxyy}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^3} - 3\frac{\widetilde{h_0}_{xx}\left(\widetilde{h_0}_{xy}^2+\widetilde{h_0}_{yy}^2\right)}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^5}.$$

Plugging all of these into the expression for $\frac{\delta\tilde{J}}{\delta\tilde{h}}\left[\widetilde{h_0}\right]$ gives us that:

$$\left.\frac{\delta\tilde{J}}{\delta\tilde{h}}\left[\widetilde{h_0}\right]\right|_{(x,y)=(x_0,y_0)} = 3\frac{\left(\widetilde{h_0}_{xx}\widetilde{h_0}_{yy}-\widetilde{h_0}_{xy}^2\right)\widetilde{h_0}_{xx}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^5} + 3\frac{\left(\widetilde{h_0}_{xx}\widetilde{h_0}_{yy}-\widetilde{h_0}_{xy}^2\right)\widetilde{h_0}_{yy}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^5}$$

$$+\frac{\widetilde{h_0}_{yyxx}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^3} - 3\frac{\widetilde{h_0}_{yy}\left(\widetilde{h_0}_{xx}^2+\widetilde{h_0}_{xy}^2\right)}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^5} - 2\frac{\widetilde{h_0}_{xyyx}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^3}$$

$$+6\frac{\widetilde{h_0}_{xy}\left(\widetilde{h_0}_{xx}\widetilde{h_0}_{xy}+\widetilde{h_0}_{yx}\widetilde{h_0}_{yy}\right)}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^5} + \frac{\widetilde{h_0}_{xxyy}}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^3} - 3\frac{\widetilde{h_0}_{xx}\left(\widetilde{h_0}_{xy}^2+\widetilde{h_0}_{yy}^2\right)}{\sqrt{1+\widetilde{h_0}_x^2+\widetilde{h_0}_y^2}^5}.$$

A much shorter expression than what we got in the middle of the proof of Theorem 5.6.2. And it is not at all a hard task to show that every term on the right-hand side of the above equation cancels out. Canceling all of the terms on the right-hand side out finally gives us that:

$$\left.\frac{\delta\tilde{J}}{\delta\tilde{h}}\left[\widetilde{h_0}\right]\right|_{(x,y)=(x_0,y_0)} = 0.$$

Thus, by the rotation trick argument we get that the functional derivative of our original functional at h_0 evaluated at x_0 is equal to zero:

$$\left.\frac{\delta J}{\delta h}[h_0]\right|_{(x,y)=(x_0,y_0)} = 0.$$

Since (x_0, y_0) was chosen arbitrarily in Ω this shows that $\frac{\delta J}{\delta h}[h_0]$ is zero everywhere:

$$\frac{\delta J}{\delta h}[h_0] \equiv 0$$

on Ω. In fact, since h_0 was chosen arbitrarily in the domain of J this shows that the functional derivative J is zero at any curve in its domain.

Look at this! with this technique we were able to shorten the calculation done in the proof of Theorem 5.6.2 by a lot. This rotation trick will be crucial in the proof of the next two theorems because proving them without this trick is next to impossible.

Section 7: Minimal Surfaces in $\mathbb{R}^n$

The next exciting variational theorem on our list is the Minimal Surface Theorem for general surface in $\mathbb{R}^n$. The question that his theorem address is exactly of the same sort that we already explored in Chapter 5 when we discussed two-dimensional minimal surfaces sitting in $\mathbb{R}^3$ except that here we do things in general n dimensions. Suppose that we have a contour in n-dimensional Euclidean space and we want to find a surface that passes through this contour and that occupies the minimum amount of surface area. Such a surface is called a minimal surface and it turns out that as in the case of two-dimensional surface, it will always have constant mean curvature equal to zero.

We will prove a form of this fact that yields to a nice and easy formulation. However, before we do that, we need to define the surface area in n-dimensions. Suppose that we have a continuously differentiable function $x_n = f(x_1, x_2, \dots, x_{n-1})$ and that Ω is a region in the x_1-x_2-…-x_{n-1} plane. Then the surface area of the surface generated by the graph of f over Ω is defined to be (here I omit the arguments of the partials of f):

$$A = \int_{\Omega} \sqrt{1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_{n-1}}\right)^2} \prod_{k=1}^{n-1} dx_k,$$

where $\int$ here denotes an integral over a region in $\mathbb{R}^n$. This is a formula that's often presented in a course on multivariable calculus and can be intuitively introduced using the Cauchy-Binet equation. Again, as we will do here, this is usually just taken to be the definition of the surface area of the graph of a function. Here I am going to use the fact that surface area is a geometric property (specifically that it is invariant under rotation) so that we can apply the trick discussed in the previous section to the proof of the Minimal Surface Theorem. I will not prove this fact here. Its proof involves a linear algebra and change of variables argument.

Now we are ready to state and prove a version of the Minimal Surface Theorem.

Theorem 6.7.1 (Graph Version of the C^2 Minimal Surface Theorem): *Let $\Omega \subseteq \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ that the Divergence Theorem applies to. Suppose that Ω is the closure of its interior and that its boundary $\partial\Omega$ is a set of zero Jordan content. Suppose also that we have a function of the form $f : \partial\Omega \to \mathbb{R}$ and that the set:*

$$\{(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n)) \in \mathbb{R}^{n+1} : (x_1, x_2, \dots, x_n) \in \partial\Omega\}$$

can be parametrized by a non-singular C^2 curve $\gamma(t)$ in $\mathbb{R}^{n+1}$. This will serve as our boundary condition for the functions in the domain of our functional. Let J be the surface area functional:

$$J[h] = \int_{\Omega} \sqrt{1 + h_{x_1}^2 + h_{x_2}^2 + \cdots + h_{x_{n-1}}^2} \prod_{k=1}^{n-1} dx_k$$

whose domain is the set of functions $h \in C^2[\bar{\Omega}]$ that satisfy the boundary conditions:

$$h(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) \quad if \quad (x_1, x_2, \dots, x_n) \in \partial\Omega$$

(in other words, the h's pass through γ over $\partial\Omega$). Now, suppose that h_0 is a local minimum of the surface area functional J. Then, the surface generated by the graph of h_0 has constantly zero mean curvature:

$$\frac{1}{n} trace\left(\frac{1}{\sqrt{1 + \sum_{k=1}^{n-1} {h_0}_{x_k}^2}} \mathcal{H}_0 \begin{bmatrix} 1 + {h_0}_{x_1}^2 & {h_0}_{x_1} {h_0}_{x_2} & \cdots & {h_0}_{x_1} {h_0}_{x_{n-1}} \\ {h_0}_{x_2} {h_0}_{x_1} & 1 + {h_0}_{x_2}^2 & \cdots & {h_0}_{x_2} {h_0}_{x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ {h_0}_{x_{n-1}} {h_0}_{x_1} & {h_0}_{x_{n-1}} {h_0}_{x_2} & \cdots & 1 + {h_0}_{x_{n-1}}^2 \end{bmatrix}^{-1}\right) \equiv 0$$

on $x \in \Omega$.[43]

Proof: We apply the rotation trick that we discussed in the previous section since J measures a geometric quantity of the h's in its domain. In this proof, let x denote the vector:

$$x = (x_1, x_2, \dots, x_{n-1}).$$

Since h_0 is a local minimum of J, the functional derivative of J at h_0 is equal to zero:

$$\frac{\delta J}{\delta h}[h_0] \equiv 0$$

on Ω. Now, choose any point $x_0 = \left({x_1}_0, {x_2}_0, \dots, {x_{(n-1)}}_0\right) \in \Omega$. We are going to show that the mean curvature of h_0 at $p_0 = \left({x_1}_0, {x_2}_0, \dots, {x_{(n-1)}}_0, h\left({x_1}_0, {x_2}_0, \dots, {x_{(n-1)}}_0\right)\right)$ is equal to zero (more precisely, we are going to show that the mean curvature of the surface generated by the graph of h_0 is equal to zero at p_0). In other words, we're concentrating at one point in Ω at a time. Rotate the surface h_0 so that the point p_0 becomes a local extremum in the manner described in the trick discussion in the previous section. The rotated version of the functional as described in the trick in the previous section here will be of the form:

[43] The same type of note applies here as in Footnote 33 in Theorem 5.5.1 where we looked at minimal surfaces in $\mathbb{R}^3$. We technically didn't define surface curvatures for surfaces generated by the graph of C^2 functions since we only defined curvatures for C^∞ surfaces. But we can extend the definition of surface curvatures to surfaces generated by C^2 functions by just defining them to be equal to the formulas that we derived for the Gaussian and mean curvatures in a graph surface parametrization at the end of Example 6.4.2. These C^2 surface curvatures also can be obtained from similar geometric interpretations as in the beginning of Section 4 in Chapter 4 and thus they are also geometric properties of C^2 surfaces (meaning that they are invariant under rotations and translations).

$$\tilde{J}[\tilde{h}] = \int_{\Omega} \sqrt{1 + \tilde{h}_{x_1}^2 + \tilde{h}_{x_2}^2 + \cdots + \tilde{h}_{x_{n-1}}^2} \prod_{k=1}^{n-1} dx_k$$

where the domain of this functional is the set of $\tilde{h} \in C^{\infty}[\tilde{\Omega}]$ that satisfy the property that all of their partial derivatives (even of order 0) match with those of $\widetilde{h_0}$ on the boundary $\partial\tilde{\Omega}$. The reason that the integrand looks the same is that we are still measuring the same property: surface area. Now by the surface version of the argument in the previous section we have that the functional derivative of $\tilde{J}$ at $\tilde{h}$ evaluated at $x = x_0$ is equal to zero since:

$$\frac{\delta \tilde{J}}{\delta \tilde{h}}\left[\widetilde{h_0}\right]\Bigg|_{x=x_0} = \frac{\delta J}{\delta h}[h_0]\Bigg|_{x=x_0} = 0.$$

Now, let's compute the functional derivative of $\tilde{J}$ at $\tilde{h}$ evaluated at $x = x_0$ explicitly. Let $\tilde{F}$ be the integrand in the integral equal to $\tilde{J}[\tilde{h}]$ above. By Definition 2.4.11 we have that the functional derivative of $\tilde{J}$ at $\tilde{h}$ evaluated at $x = x_0$ is given by: (here the partials of $\tilde{F}$ are being evaluated at $\left(x_1, x_2, \dots, x_{n-1}, \widetilde{h_0}, \widetilde{h_0}_{x_1}, \widetilde{h_0}_{x_2}, \dots, \widetilde{h_0}_{x_{n-1}}\right)$):

$$\frac{\delta \tilde{J}}{\delta \tilde{h}}\left[\widetilde{h_0}\right]\Bigg|_{x=x_0} = \left(\frac{\partial F}{\partial h} - \sum_{k=1}^{n-1} \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial h_{x_1}}\right)\right)\Bigg|_{x=x_0}$$

$$= \left(-\sum_{k=1}^{n-1} \frac{\partial}{\partial x_k}\left(\frac{h_{x_k}}{\sqrt{1 + \tilde{h}_{x_1}^2 + \tilde{h}_{x_2}^2 + \cdots + \tilde{h}_{x_{n-1}}^2}}\right)\right)\Bigg|_{x=x_0}.$$

Calculating the above $\frac{\partial}{\partial x_k}$ partials, while taking into account that $\widetilde{h_0}_{x_1}, \widetilde{h_0}_{x_2}, \dots, \widetilde{h_0}_{x_{n-1}}$ are all zero at $x = x_0$ since x_0 is a local extremum of $\widetilde{h_0}$, gives that (here all of the partials of $\widetilde{h_0}$ are being evaluated at $x = x_0$):

$$\frac{\delta \tilde{J}}{\delta \tilde{h}}\left[\widetilde{h_0}\right]\Bigg|_{x=x_0} = -\sum_{k=1}^{n-1} \widetilde{h_0}_{x_k x_k}(x_0) = -trace\left(\widetilde{\mathcal{H}_0}(x_0)\right)$$

where $\widetilde{\mathcal{H}_0}$ denotes the Hessian matrix of $\widetilde{h_0}$. So, since the functional derivative of $\tilde{J}$ at $\tilde{h}$ evaluated at $x = x_0$ is equal to zero the above equation finally implies that:

$Equation$ 6.7.2: $$trace\left(\widetilde{\mathcal{H}_0}(x_0)\right) = 0.$$

Now comes the heart of the matter: the left-hand side of the above equation turns out to be equal to n times the mean curvature of the surface $\widetilde{h_0}$ at p_0 which is furthermore equal to n times the mean curvature of h_0 at p_0 since mean curvature is a geometric property and thus invariant under rotation. Let's prove that this indeed the case. Let $\widetilde{H_0}(x)$ denote the mean curvature of $\widetilde{h_0}$ at above x (at the point $\left(x_1, x_2, \dots, x_{n-1}, h(x_1, x_2, \dots, x_{n-1})\right)$ that is). Plugging everything here into the formula for the mean curvature of a graph that we derived in Example 6.4.2 gives us that the mean curvature of $\widetilde{h_0}$ at p_0 is given by:

$$\widetilde{H_0}(x_0) = \frac{1}{n} trace\left(\frac{1}{\sqrt{1+\sum_{k=1}^{n-1}\widetilde{h_0}_{x_k}^2(x_0)}}\widetilde{\mathcal{H}_0}(x_0)\left(M_\beta(x_0)\right)^{-1}\right)$$

where the matrix M_β is given by:

$$M_\beta = \begin{bmatrix} 1+\widetilde{h_0}_{x_1}^2 & \widetilde{h_0}_{x_1}\widetilde{h_0}_{x_2} & \cdots & \widetilde{h_0}_{x_1}\widetilde{h_0}_{x_{n-1}} \\ \widetilde{h_0}_{x_2}\widetilde{h_0}_{x_1} & 1+\widetilde{h_0}_{x_2}^2 & \cdots & \widetilde{h_0}_{x_2}\widetilde{h_0}_{x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{h_0}_{x_{n-1}}\widetilde{h_0}_{x_1} & \widetilde{h_0}_{x_{n-1}}\widetilde{h_0}_{x_2} & \cdots & 1+\widetilde{h_0}_{x_{n-1}}^2 \end{bmatrix}.$$

Since x_0 is a local extremum of $\widetilde{h_0}$, all of the first order partials of $\widetilde{h_0}$ in the above expression for the mean curvature $\widetilde{H_0}(x_0)$ go away. For example, the denominator $\sqrt{1+\sum_{k=1}^{n-1} h_{x_k}^2(x_0)}$ just becomes 1 and $M_\beta(x_0)$ just becomes the identity matrix. So the above expression for the mean curvature of $\widetilde{h_0}$ at p_0 can be rewritten as:

$$\widetilde{H_0}(x_0) = \frac{1}{n} trace\left(\widetilde{\mathcal{H}_0}(x_0)\right).$$

Thus Equation 6.7.2 can indeed be rewritten as:

$$n\widetilde{H_0}(x_0) = 0,$$

which trivially implies that:

$$\widetilde{H_0}(x_0) = 0.$$

Since mean curvature is invariant under rotations, we get that the mean curvature of our original surface h_0 at p_0 is also equal to zero (here H_0 denotes the mean curvature of h_0):

$$H_0(x_0) = 0.$$

Since x_0 was an arbitrarily chosen point in Ω, this finally proves that the mean curvature of h_0 is constant zero everywhere:

$$H_0 \equiv 0.$$

With this we have proved the theorem!

■

With the above theorem we have proved a special case of the statement that all minimal surfaces in $\mathbb{R}^n$ have constant mean curvature equal to zero. An amazing fact!

I think it's important to go back and see where exactly the trick of rotating the surfaces in the domain of the functional helped in the above proof. Evaluating the functional's derivative at extremum points by rotating the surface each time made all of the first partials of the surface in the expression for the functional derivative go away. This for example made the equation for the functional derivative being equal to zero at a point p_0 in the above proof a very short and elegant equation, a mere:

$$trace\left(\widetilde{\mathcal{H}_0}(x_0)\right) = 0.$$

And the most important task of showing that the left-hand side of the above equation is in fact equal to n times the mean curvature of the surface $\widetilde{h_0}$ at p_0, which implied that the mean curvature there was zero, was also really easy because at an extremum point the equation for the mean curvature is the short equation:

$$\widetilde{H_0}(x_0) = \frac{1}{n} trace\left(\widetilde{\mathcal{H}_0}(x_0)\right).$$

Had we chosen not to work with extremum points, then the task of showing that the functional derivative at x_0 was equal to n times the mean curvature of h_0 over x_0 would have boiled down to proving the equation (here the partials of h_0 and $\mathcal{H}_0$ [the Hessian of h_0] are being evaluated at x_0):

$$-\sum_{k=1}^{n-1}\left(\frac{h_{0_{x_k}}}{\sqrt{1+\sum_{k=1}^{n-1} h_{0_{x_k}}^{\ 2}}}\right) + \sum_{k=1}^{n-1}\frac{h_{0_{x_k}}\sum_{j=1}^{n} h_{0_{x_j}} h_{0_{x_j x_k}}}{\sqrt{1+\sum_{k=1}^{n-1} h_{0_{x_k}}^{\ 2}}^{\,3}}$$

$$= \frac{1}{n} trace\left(\frac{1}{\sqrt{1+\sum_{k=1}^{n-1} h_{0_{x_k}}^{\ 2}}}\mathcal{H}_0\begin{bmatrix} 1+h_{0_{x_1}}^{\ 2} & h_{0_{x_1}}h_{0_{x_2}} & \cdots & h_{0_{x_1}}h_{0_{x_{n-1}}} \\ h_{0_{x_2}}h_{0_{x_1}} & 1+h_{0_{x_2}}^{\ 2} & \cdots & h_{0_{x_2}}h_{0_{x_{n-1}}} \\ \vdots & \vdots & \ddots & \vdots \\ h_{0_{x_{n-1}}}h_{0_{x_1}} & h_{0_{x_{n-1}}}h_{0_{x_2}} & \cdots & 1+h_{0_{x_{n-1}}}^{\ 2} \end{bmatrix}^{-1}\right).$$

An incredibly scary equation, which shows what we evaded by employing the above rotation trick. I think that the rotation trick made the above proof of the Minimal Surface Theorem really elegant.

Section 8: The Global Gauss-Bonnet Theorem in $\mathbb{R}^n$

We finally get to what I consider the culmination of all of the mathematics that we have been doing so far in this book. We will prove a version of a corollary of the Global Gauss-Bonnet Theorem in $\mathbb{R}^n$. This theorem that we will proving here stems from the same exact set of ideas that we had in Section 6 of Chapter 5. The fact that we will be proving here is that if you take a closed surface in $\mathbb{R}^n$ that can be obtained by smoothly deforming a fundamental surface, such as an $(n-1)$-dimensional sphere or an $(n-1)$-dimensional torus, then the total Gaussian curvature of your surface will be the same as the total Gaussian curvature of that fundamental surface. Here the **total Gaussian curvature** of a surface S in $\mathbb{R}^n$ is defined, just like in Chapter 5, as the integral of the Gaussian curvature over the entire surface:

$$K[S] = \oint_S K \, d\sigma$$

where $\oint$ here denotes a surface integral and K is the Gaussian curvature as a function of the points on the surface. There is one point of technicality that needs to be mentioned here that I've been sweeping under the rug for some time, and that is the subject of the surface curvatures being well defined as we do these surface integrals. When we write the above surface integral, there is a possible conflict in that as we integrate over the surface we don't know which Gaussian curvature to use at each point since there are possibly two at that point that differ in sign. As mentioned before, if the dimension of the surface is even then the Gaussian curvature is always well defined at each point and there is no problem (as was the case in Section 6 of Chapter 5). However, if the dimension of the surface is odd then at each point the Gaussian curvature is well defined only up to sign and that sign turns out to depend on which direction the unit normal to the surface at each point points in. In that case, in order to make the above surface integral make sense we have to set a convention about which direction we always make the unit normal to the surface point in, and that convention is often to let the unit normal to point out of the region enclosed by the surface. In other words, when we construct the Gauss map for our surface parametrizations in order to calculate the Gaussian curvature let the Gauss map point away from the region enclosed, which you can accomplish by multiplying it by -1 if it isn't already doing so. That way the above integral makes sense.

Technically the above well-defined issue needed to be addressed in the minimal surface sections as well since mean curvature is *always* well defined only up to sign. But there it didn't really present much of a problem because in that section we ended up proving that minimal surfaces have constant zero mean curvature and $0 = -0$.

The type of surface deformations that we will be dealing with here are the type that locally deform surface, just like we had in Chapter 5. So let us take the definition that we had for local surface deformations that we had in Section 6 of the previous chapter and generalize it to general surface in $\mathbb{R}^n$.

Definition 6.8.1: *Let S be a smooth $(n-1)$-dimensional surface sitting in $\mathbb{R}^n$ and $p \in S$ be any point on it. In an open neighborhood V of p, S can be represented as the graph of a C^∞ function of the form $f : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}^{n-1}$ is some open set. Let's suppose that f is of the form $x_n = f(x_1, x_2, \dots, x_{n-1})$ (this definition is similar in the other cases mentioned in Definition 6.2.1). Let's also suppose that our open set V is of the form $U \times (\alpha, \beta)$ (a set theoretic cylinder), which can always be arranged by choosing our U and (α, β) small enough.*

Now, let p_π be the projection of p onto the x_1-x_2-…-x_{n-1} plane. Since $p_\pi \in U$ and U is open, there exists an open ball $B_r(p_\pi)$ centered at p_π such that $B_r(p_\pi) \subseteq U$. Ok, let $g \in C^\infty[U]$ be any function such that g vanishes outside of $B_{r/2}(p_\pi)$ (note the $r/2$). Let $\Lambda : U \times [0, a] \to \mathbb{R}$ be the ∞-smooth linear flow defined by:

$$\Lambda(x_1, x_2, \dots, x_{n-1}, t) = f(x_1, x_2, \dots, x_{n-1}) + g(x_1, x_2, \dots, x_{n-1})t.$$

where for each $t \in [0, a]$ *the graph of* $\Lambda(x_1, x_2, \dots, x_{n-1})$ *is inside of* V. *Finally, for each time* $t \in [0, a]$ *let* $\mathcal{S}(t)$ *be the surface defined by:*

$$\mathcal{S}(t) = (S \setminus V)$$
$$\cup \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : (x_1, x_2, \dots, x_{n-1}) \in U \quad and \quad x_n = \Lambda(x_1, x_2, \dots, x_{n-1}, t)\}.$$

The function $\mathcal{S}(t)$ *is called a* ***smooth local graph deformation*** *of* S. *The surface* $\mathcal{S}(a)$ *is the surface that* $S = \mathcal{S}(0)$ *deforms to under this deformation.*[44]

The proof of the fact that such an $a > 0$ always exists to make $\Lambda(x_1, x_2, \dots, x_{n-1}, t)$ stay inside of V is similar to the analogous proof in the three-dimensional case that we gave in the discussion following Definition 5.6.1. I will leave the details to the reader. Now we are ready to prove the theorem that says that the total Gaussian curvature of a surface is invariant under such smooth local graph deformations.

Theorem 6.8.2 (Invariance of the Total Gaussian Curvature under Smooth Local Graph Deformations in $\mathbb{R}^n$): *Suppose that* S *is a smooth* $(n-1)$*-dimensional surface sitting in* $\mathbb{R}^n$ *and that* $\mathcal{S}(t)$ *is a smooth local graph deformation of* S *defined over the interval* $t \in [0, a]$ *where* $a > 0$. *Then, for any* $t \in [0, a]$, *the total Gaussian curvature of* $\mathcal{S}(t)$ *is equal to:*

$$K[\mathcal{S}(t)] = K[S].$$

In other words, deforming a surface using smooth local graph deformations doesn't change its total Gaussian curvature.

Proof: Let's suppose that our local graph deformation $\mathcal{S}(t)$ is of the form:

$$\mathcal{S}(t) = (S \setminus V)$$
$$\cup \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : (x_1, x_2, \dots, x_{n-1}) \in U \quad and \quad x_n = \Lambda(x_1, x_2, \dots, x_{n-1}, t)\}$$

(this proof in the other cases mentioned Definition 6.2.1 for the forms of f is similar). Let's carry over all of the notation that we had Definition 6.8.1 into this proof. In other words, let $B_r(p_\pi) \subseteq U$, $f, g \in C^\infty[U]$ such that g vanishes outside of $B_{r/2}(p_\pi)$, and:

$$\Lambda(x_1, x_2, \dots, x_{n-1}, t) = f(x_1, x_2, \dots, x_{n-1}) + g(x_1, x_2, \dots, x_{n-1})t.$$

At $t = 0$, it's obvious that $K[\mathcal{S}(0)] = K[S]$ since $\mathcal{S}(0) = S$. So let's prove that $K[\mathcal{S}(t)]$ is constantly equal to $K[S]$ by just showing that its time derivative is constantly equal to zero. We have that:

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \frac{d}{dt}\left(\oint_{\mathcal{S}(t)} K \, d\sigma\right).$$

[44] An n-dimensional version of Footnote 34 on page 194 here as well.

Outside of V our surface is not deforming in any way and so the total Gaussian curvature there is unchanging. In other words, since there is no deformation happening outside of V we have that:

$$\frac{d}{dt}\left(\oint_{\mathcal{S}(t)\setminus V} K\,d\sigma\right) = 0.$$

So really, we can just rewrite the expression for the time derivative of $K[\mathcal{S}(t)]$ as:

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \frac{d}{dt}\left(\oint_{\mathcal{S}(t)\setminus V} K\,d\sigma + \oint_{\mathcal{S}(t)\cap V} K\,d\sigma\right) = \frac{d}{dt}\left(\oint_{\mathcal{S}(t)\cap V} K\,d\sigma\right).$$

This is nice because now we expressed the time derivative of $K[\mathcal{S}(t)]$ as an integral over a local piece of the surface which by the way can be represented as the graph of the function $\Lambda(x_1, x_2, \dots, x_{n-1}, t)$ for every fixed t. In explanation, the piece of the surface $\mathcal{S}(t) \cap V$ is the graph of $\Lambda(x_1, x_2, \dots, x_{n-1}, t)$ over U and so we can rewrite the last surface integral in the above expression as (here I use the Newtonian notation for partial derivatives):

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \frac{d}{dt}\left(\int_U K \sqrt{1 + \sum_{m=1}^{n-1}\left(\Lambda_{x_m}(x_1, x_2, \dots, x_{n-1}, t)\right)^2} \prod_{m=1}^{n-1} dx_m\right).$$

Let $\mathcal{H}_{\Lambda(x_1,x_2,\dots,x_{n-1},t)}(x_1, x_2, \dots, x_{n-1})$, or simply $\mathcal{H}_\Lambda(x_1, x_2, \dots, x_{n-1})$ is we don't want to write the arguments of Λ, denote the Hessian matrix for $\Lambda(x_1, x_2, \dots, x_{n-1}, t)$ as a function of $(x_1, x_2, \dots, x_{n-1})$ for each fixed t. By the formula for the Gaussian curvature of a graph that we derived in Example 6.4.2, we get that the above integral can be rewritten as (here I omit the argument of Λ and $\mathcal{H}_\Lambda$; they are being evaluated at $(x_1, x_2, \dots, x_{n-1}, t)$ and $(x_1, x_2, \dots, x_{n-1})$ respectively):

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \frac{d}{dt}\left(\int_U \frac{\det(\mathcal{H}_\Lambda)}{\sqrt{1 + \sum_{m=1}^{n-1}\Lambda_{x_m}^2}^{\,n+1}} \sqrt{1 + \sum_{m=1}^{n-1}\Lambda_{x_m}^2} \prod_{m=1}^{n-1} dx_m\right)$$

$$= \frac{d}{dt}\left(\int_U \frac{\det(\mathcal{H}_\Lambda)}{\sqrt{1 + \sum_{m=1}^{n-1}\Lambda_{x_m}^2}^{\,n}} \prod_{m=1}^{n-1} dx_m\right).$$

Since we're going to carry the derivative under the integral sign we, it would be nice to have our domain of integration compact. Notice that the surface does not change outside of the ball $B_{3r/4}(p_\pi)$ since g vanishes outside of $B_{r/2}(p_\pi)$ ($3r/4$ is just some number I chose between $r/2$

and r; we'll soon see why we need such a radius). So the above expression is in fact equivalent to:

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \frac{d}{dt}\left(\int\limits_{\overline{B_{3r/4}(p_\pi)}} \frac{\det(\mathcal{H}_\Lambda)}{\sqrt{1+\sum_{m=1}^{n-1} \Lambda_{x_m}^2}^{\,n}} \prod_{m=1}^{n-1} dx_m \right).$$

Now that we have a compact region of integration, we will be able to carry the derivative under the integral sign. We will however carry the derivative under the integral sign covertly by using Theorem 3.7.5 that already does this for us. Let us form the following local "total Gaussian curvature functional:"

$$J[h] = \int\limits_{\overline{B_{3r/4}(p_\pi)}} \frac{\det(\mathcal{H}_h)}{\sqrt{1+\sum_{m=1}^{n-1} h_{x_m}^2}^{\,n}} \prod_{m=1}^{n-1} dx_m,$$

where $\mathcal{H}_h$ is the Hessian matrix for h (evaluated at $(x_1, x_2, \ldots, x_{n-1})$), defined over the space of functions $h \in C^\infty\left[\overline{B_{r/2}(p_\pi)}\right]$ that satisfy the boundary conditions of being equal to $f(x_1, x_2, \ldots, x_{n-1})$ on the boundary $\partial B_{r/2}(p_\pi)$ and that all their first order partials are equal to that of $f(x_1, x_2, \ldots, x_{n-1})$ on this boundary. Notice that we can now reformulate the equation above for the time derivative of $K[\mathcal{S}(t)]$ as:

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \frac{d}{dt}(J[\Lambda]).$$

It's easy to check that the restriction of Λ to $\overline{B_{r/2}(p_\pi)}$ is a flow that stays inside of the domain J (to see this just relook at the equation for Λ in terms of f and g again). But wait, we know an expression for the quantity on the right because we derived it in Theorem 3.7.5. By Theorem 3.7.5 we have that:

Equation 6.8.3:
$$\frac{d}{dt}(K[\mathcal{S}(t)]) = \int\limits_{\overline{B_{3r/4}(p_\pi)}} \frac{\delta J}{\delta h}[\Lambda]\Lambda_t \prod_{m=1}^{n-1} dx_m.$$

So if we show that $\frac{\delta J}{\delta h}[\Lambda] \equiv 0$ over $t \in [0, a]$, then we will get that the above equation implies that the time derivative of $K[\mathcal{S}(t)]$ is also constantly equal to zero. So let's show that $\frac{\delta J}{\delta h} \equiv 0$. Let's calculate the variational derivative of J at any general h_0 sitting inside of this functional's domain. Total Gaussian curvature is a geometric property since it is the surface integral of the Gaussian curvature and both surface integrals and Gaussian curvature are geometric quantities (specifically they are invariant under rotation). Thus our functional J above is measuring a geometric quantity of the surfaces in its domain. So we can apply the trick discussed in Section 6 here to evaluate the functional derivative of J at h_0. In what follows, let x denote the vector:

$$x = (x_1, x_2, \ldots, x_{n-1}).$$

Now, pick any point $x_0 \in \left(x_{1_0}, x_{2_0}, \dots, x_{(n-1)_0}\right) \in \overline{B_{3r/4}(p_\pi)}$. We are going to show that the functional derivative of J at h_0 evaluated at $x = x_0$ is equal to zero. Rotate the surface h_0 so that the point:

$$p_0 = \left(x_{1_0}, x_{2_0}, \dots, x_{(n-1)_0}, h_0\left(x_{1_0}, x_{2_0}, \dots, x_{(n-1)_0}\right)\right)$$

becomes a local extremum in the manner described in the trick discussion in Section 6.[45] The rotated version of the functional as described in the trick in Section 6 here will be of the form:

$$\tilde{J}[\tilde{h}] = \int_{\tilde{\Omega}} \frac{\det(\mathcal{H}_{\tilde{h}})}{\sqrt{1 + \sum_{m=1}^{n-1} \tilde{h}_{x_m}^2}^{\,n}} \prod_{m=1}^{n-1} dx_m,$$

where $\mathcal{H}_{\tilde{h}}$ is the Hessian matrix for $\tilde{h}$ and the domain of this functional is the set of $\tilde{h} \in C^\infty[\tilde{\Omega}]$ that satisfy the property that all of their partial derivatives (even of order 0) match with those of $\widetilde{h_0}$ on the boundary $\partial\tilde{\Omega}$. The reason that the integrand looks the same is that we are still measuring the same property: total Gaussian curvature. Now, calculate the functional derivative of this $\tilde{J}$ at $\widetilde{h_0}$ evaluated at $x = x_0$. Let $\tilde{F}$ be the integrand of the integral equal to $\tilde{J}[\tilde{h}]$ above. Then we have that (here $\tilde{F}$ is being evaluated [in indexed argument notation] at $\left(\{x_k\}_{k=1}^{n-1}, \widetilde{h_0}, \left\{\widetilde{h_0}_{x_k}\right\}_{k=1}^{n-1}, \left\{\left\{\widetilde{h_0}_{x_k x_j}\right\}_{k=1}^{n-1}\right\}_{j=k}^{n-1}\right)$):

$$\left.\frac{\delta \tilde{J}}{\delta \tilde{h}}\left[\widetilde{h_0}\right]\right|_{x=x_0} = \left.\left(\frac{\partial \tilde{F}}{\partial \tilde{h}} - \sum_{k=1}^{n-1} \frac{\partial}{\partial x_k}\left(\frac{\partial \tilde{F}}{\partial \tilde{h}_{x_k}}\right) + \sum_{k=1}^{n-1}\sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_k \partial x_j}\left(\frac{\partial \tilde{F}}{\partial \tilde{h}_{x_k x_j}}\right)\right)\right|_{x=x_0}.$$

Let's prove that the right-hand side is equal to zero. If A is matrix, let $A_{k,j}$ denote the matrix obtained by removing the k^{th} row and j^{th} column. Now, calculating the above partials of $\tilde{F}$ gives us that the right-hand side of the above equation is equal to:

$$\left.\left(-\sum_{k=1}^{n-1} \frac{\partial}{\partial x_k}\left(-n \frac{\tilde{h}_{x_k} \det(\mathcal{H}_{\tilde{h}})}{\sqrt{1 + \sum_{m=1}^{n-1} \tilde{h}_{x_m}^2}^{\,n+2}}\right) + \sum_{k=1}^{n-1}\sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_k \partial x_j}\left(\frac{(-1)^{k+j} \det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)}{\sqrt{1 + \sum_{m=1}^{n-1} \tilde{h}_{x_m}^2}^{\,n}}\right)\right)\right|_{x=x_0}.$$

Calculating the above partials, while remembering that $\widetilde{h_0}_{x_1}, \widetilde{h_0}_{x_2}, \dots, \widetilde{h_0}_{x_{n-1}}$ at $x = x_0$ are all equal to zero since x_0 is a local extremum of $\widetilde{h_0}$, gives us that the previous equation becomes (I omit the argument of $\mathcal{H}_{\tilde{h}}$ and $\tilde{h}$ here, they are both being evaluated at $x = x_0$):

[45] Here however when you do the rotation procedure you will have to use a C^∞ function φ (see Section 6) rather than a C^2 one since we are dealing with C^∞ h's.

$Equation\ 6.8.4$: $$\frac{\delta \tilde{J}}{\delta \tilde{h}}\left[\widetilde{h_0}\right]\Bigg|_{x=x_0} = n\sum_{k=1}^{n-1}\left(\frac{\tilde{h}_{x_k x_k}\det(\mathcal{H}_{\tilde{h}})}{\sqrt{1+\sum_{m=1}^{n-1}\tilde{h}_{x_m}^2}^{\,n+2}}\right)$$

$$+\sum_{k=1}^{n-1}\sum_{j=1}^{n-1}\left(\frac{(-1)^{k+j}\frac{\partial^2}{\partial x_k \partial x_j}\left(\det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)\right)}{\sqrt{1+\sum_{m=1}^{n-1}\tilde{h}_{x_m}^2}^{\,n}} - n\frac{(-1)^{k+j}\det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)\sum_{m=1}^{n-1}\left(\tilde{h}_{x_m x_k}\tilde{h}_{x_m x_j}\right)}{\sqrt{1+\sum_{m=1}^{n-1}\tilde{h}_{x_m}^2}^{\,n+2}}\right).$$

Let’s show that the first and the third terms on the right-hand side cancel out. In other words, let us show that:

$$n\sum_{k=1}^{n-1}\left(\frac{\tilde{h}_{x_k x_k}\det(\mathcal{H}_{\tilde{h}})}{\sqrt{1+\sum_{m=1}^{n-1}\tilde{h}_{x_m}^2}^{\,n+2}}\right) = n\sum_{k=1}^{n-1}\sum_{j=1}^{n-1}\left(\frac{(-1)^{k+j}\det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)\sum_{m=1}^{n-1}\left(\tilde{h}_{x_m x_k}\tilde{h}_{x_m x_j}\right)}{\sqrt{1+\sum_{m=1}^{n-1}\tilde{h}_{x_m}^2}^{\,n+2}}\right).$$

Canceling n and $\sqrt{1+\sum_{m=1}^{n-1}\tilde{h}_{x_m}^2}^{\,n+2}$ from both sides gives us that this is equivalent to showing that:

$Equation\ 6.8.5$:

$$\sum_{k=1}^{n-1}\left(\tilde{h}_{x_k x_k}\det(\mathcal{H}_{\tilde{h}})\right) = \sum_{k=1}^{n-1}\sum_{j=1}^{n-1}\left((-1)^{k+j}\det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)\sum_{m=1}^{n-1}\left(\tilde{h}_{x_m x_k}\tilde{h}_{x_m x_j}\right)\right).$$

Tucking in $(-1)^{k+j}\det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)$ on the right-hand side into the inner most sum gives us that the right-hand side of the above equation is equal to (in the first equality below I interchange summations):

$$\sum_{k=1}^{n-1}\sum_{j=1}^{n-1}\sum_{m=1}^{n-1}\left((-1)^{k+j}\det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)\tilde{h}_{x_m x_k}\tilde{h}_{x_m x_j}\right)$$

$$=\sum_{k=1}^{n-1}\sum_{m=1}^{n-1}\sum_{j=1}^{n-1}\left((-1)^{k+j}\det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)\tilde{h}_{x_m x_k}\tilde{h}_{x_m x_j}\right)$$

$$=\sum_{k=1}^{n-1}\sum_{m=1}^{n-1}\left(\tilde{h}_{x_m x_k}\sum_{j=1}^{n-1}\left((-1)^{k+j}\det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)\tilde{h}_{x_m x_j}\right)\right).$$

Now, the inner sum $\sum_{j=1}^{n-1}\left((-1)^{k+j}\det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)\tilde{h}_{x_m x_j}\right)$ is equal to the determinant of the matrix $\mathcal{H}$ is its k^{th} row replaced with the row $\left[\tilde{h}_{x_m x_1} \quad \tilde{h}_{x_m x_2} \quad \cdots \quad \tilde{h}_{x_m x_{n-1}}\right]$. But such a matrix has zero determinant if $m \neq k$ because in such a case then the m^{th} row and k^{th} row are equal and thus linearly dependent. So the terms in the above sum where $m \neq k$ go away and for the terms where $m = k$, we have that $\sum_{j=1}^{n-1}\left((-1)^{k+j}\det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)\tilde{h}_{x_m x_j}\right) = \det(\mathcal{H}_{\tilde{h}})$. So the quantity on the right-hand side of Equation 6.8.5 becomes:

$$\sum_{k=1}^{n-1}\left(\tilde{h}_{x_k x_k}\det(\mathcal{H}_{\tilde{h}})\right),$$

which is the left-hand side of Equation 6.8.5. Thus Equation 6.8.5 holds and so the first and the third terms on the right-hand side of Equation 6.8.4 cancel out. So, we have that Equation 6.8.4 becomes:

$$\textit{Equation } 6.8.6: \qquad \frac{\delta\tilde{J}}{\delta\tilde{h}}\left[\widetilde{h_0}\right]\Bigg|_{x=x_0} = \sum_{k=1}^{n-1}\sum_{j=1}^{n-1}\frac{(-1)^{k+j}\dfrac{\partial^2}{\partial x_k \partial x_j}\left(\det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)\right)}{\sqrt{1+\sum_{m=1}^{n-1}\tilde{h}_{x_m}^2}^{\,n}}.$$

One final thing to do. We have to show that the sum on the right-hand side is equal to 0. This is equivalent to showing that the following sum is equal to zero:

$$\textit{Equation } 6.8.7: \qquad \sum_{k=1}^{n-1}\sum_{j=1}^{n-1}\left((-1)^{k+j}\frac{\partial^2}{\partial x_k \partial x_j}\left(\det\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)\right)\right) = 0.$$

Calculating the $\partial^2/\partial x_k \partial x_j$ partial in the above sum gives that the quantity on the left-hand side of the above equation becomes:

$\textit{Equation}$ 6.8.8:

$$\sum_{k=1}^{n-1}\sum_{j=1}^{n-1}\sum_{\substack{r=1\\ r\neq k}}^{n-1}\sum_{\substack{s=1\\ s\neq j}}^{n-1}\left((-1)^{\sigma_{k,j,r,s}}\frac{\partial^4\tilde{h}}{\partial x_r \partial x_s \partial x_k \partial x_j}\det\left(\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)_{r,s}\right)\right)$$

$$+\sum_{k=1}^{n-1}\sum_{j=1}^{n-1}\sum_{\substack{r=1\\ r\neq k}}^{n-1}\sum_{\substack{s=1\\ s\neq j}}^{n-1}\sum_{\substack{d=1\\ d\neq k\\ d\neq r}}^{n-1}\sum_{\substack{v=1\\ v\neq j\\ v\neq s}}^{n-1}\left((-1)^{\tau_{k,j,r,s,d,v}}\frac{\partial^3\tilde{h}}{\partial x_r \partial x_s \partial x_k}\frac{\partial^3\tilde{h}}{\partial x_d \partial x_v \partial x_j}\det\left(\left(\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)_{r,s}\right)_{d,v}\right)\right),$$

where $\sigma_{k,j,r,s}$ and $\tau_{k,j,r,s,d,v}$ are integers whose purpose in the above sums is to give the correct sign when -1 is raised to these numbers. It is rather cumbersome to write an explicit equation for these $\sigma_{k,j,r,s}$'s and $\tau_{k,j,r,s,d,v}$'s. Now, we have to show that both of the above sums are equal to zero. Let's first look at the first sum in the above expression. For every fixed integers k, j, r, s, the two terms (the following two terms differ because I interchanged places of k and r):

$$(-1)^{\sigma_{k,j,r,s}} \frac{\partial^4 \tilde{h}}{\partial x_r \partial x_s \partial x_k \partial x_j} \det\left(\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)_{r,s}\right),$$

$$(-1)^{\sigma_{r,j,k,s}} \frac{\partial^4 \tilde{h}}{\partial x_k \partial x_s \partial x_r \partial x_j} \det\left(\left((\mathcal{H}_{\tilde{h}})_{r,j}\right)_{k,s}\right),$$

have the same magnitude (absolute value). However, a basic matrix analysis on the matrix $\mathcal{H}_{\tilde{h}}$ will show that sign of $(-1)^{\sigma_{k,j,r,s}}$ and $(-1)^{\sigma_{r,j,k,s}}$ differ by a negative sign. This comes from basic row counting principles, which is used in the proof of the fact that interchanging two rows in a matrix changes the sign of its determinant where its determinant is defined through the cofactor expansion formula. So these two terms in the above sum cancel each other out. Since every term in the first sum in Equation 6.8.8 can be paired off in this manner, we get that the first sum in Equation 6.8.8 is equal to zero.

Now let's show that the second sum in Equation 6.8.8 is equal to zero. For every fixed integers k, j, r, s, the two terms (the following two terms differ because I interchanged places of r and d):

$$(-1)^{\tau_{k,j,r,s,d,v}} \frac{\partial^3 \tilde{h}}{\partial x_r \partial x_s \partial x_k} \frac{\partial^3 \tilde{h}}{\partial x_d \partial x_v \partial x_j} \det\left(\left(\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)_{r,s}\right)_{d,v}\right),$$

$$(-1)^{\tau_{k,j,d,s,r,v}} \frac{\partial^3 \tilde{h}}{\partial x_d \partial x_s \partial x_k} \frac{\partial^3 \tilde{h}}{\partial x_r \partial x_v \partial x_j} \det\left(\left(\left((\mathcal{H}_{\tilde{h}})_{k,j}\right)_{d,s}\right)_{r,v}\right),$$

have the same magnitude (absolute value). However, by a similar a basic matrix analysis on the matrix $(\mathcal{H}_{\tilde{h}})_{k,j}$ will show that sign of $(-1)^{\tau_{k,j,r,s,d,v}}$ and $(-1)^{\tau_{k,j,d,s,r,v}}$ differ by a negative sign. So these two terms in the second sum cancel each other out. Just as with the first sum, since every term in the second sum in Equation 6.8.8 can be paired off in this manner, we get that the second sum in Equation 6.8.8 is equal to zero. So we finally get that Equation 6.8.8 holds and thus Equation 6.8.6 becomes:

$$\frac{\delta \tilde{J}}{\delta \tilde{h}}\left[\widetilde{h_0}\right]\bigg|_{x=x_0} = 0.$$

Awesome! By the rotation argument, we have that this functional derivative is equal to the functional derivative of our original J at h_0 evaluated at $x = x_0$:

$$\frac{\delta J}{\delta h}[h_0]\bigg|_{x=x_0} = 0.$$

Since $x_0 \in \Omega$ was chosen arbitrarily, this shows that the functional derivative of J at h_0 is constantly equal to zero:

$$\frac{\delta J}{\delta h}[h_0] \equiv 0$$

on Ω. Furthermore, since h_0 was arbitrarily chosen in the domain of J this shows that the functional derivative of J at any of the surfaces in its domain is equal to zero. Particularly, the functional derivative of J at $\Lambda(x_1, x_2, \dots, x_{n-1}, t)$ for every fixed t is equal to zero. Plugging this fact into Equation 6.8.3 finally gives us that:

$$\frac{d}{dt}(K[\mathcal{S}(t)]) = 0$$

and so $K[\mathcal{S}(t)]$ is constantly $K[S]$ for all $t \in [0, a]$.

■

The above result shows that the total Gaussian curvature of a surface is invariant under smooth local graph deformations of surfaces. A powerful result because it says that if you take a fundamental surface, such as the sphere or torus, and apply many smooth local graph deformations to it, then its total Gaussian curvature will remain unchanged.

What is, for example, the total Gaussian curvature of a surface S that can be obtained by multiple applications of smooth local graph deformation to the sphere of radius $r > 0$? Well, it's not hard to show that the Gaussian curvature of an $(n-1)$-dimensional sphere $\partial B_r(0)$ of radius $r > 0$ sitting in $\mathbb{R}^n$ is constantly $1/r^{n-1}$ everywhere (I will leave the verification of this to the reader). A well-known formula for the surface area of an $(n-1)$-dimensional sphere sitting in $\mathbb{R}^n$ is given by:

$$A[\partial B_r(0)] = \begin{cases} \dfrac{\sqrt{2\pi}^n}{\prod_{k=1}^{(n-2)/2} 2k} r^{n-1} & if \quad n \ is \ even \\ \dfrac{2\sqrt{2\pi}^{n-1}}{\prod_{k=0}^{(n-1)/2} 2k+1} r^{n-1} & if \quad n \ is \ odd \end{cases},$$

which is often written in the compact notation:

$$A[\partial B_r(0)] = \frac{2\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2}\right)} r^{n-1}$$

where Γ is the Gamma function (this equation is not hard to derive; it is just a direct application of integration). Then, by the above theorem we get that the total Gaussian curvature of S is given by:

$$K[S] = K[\partial B_r(0)] = \oint_{\partial B_r(0)} K \, d\sigma = \oint_{\partial B_r(0)} \frac{1}{r^{n-1}} d\sigma = \frac{1}{r^{n-1}} \oint_{\partial B_r(0)} d\sigma = \frac{1}{r^{n-1}} \frac{2\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2}\right)} r^{n-1}$$
$$= \frac{2\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2}\right)}.$$

Thus, the total Gaussian curvature of any such surface S is equal to $\frac{2\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2}\right)}$.

Section 9: Concluding Words

Whether you're here because you have reached the end of the book or you flipped to the end to see how it ends (I admit that I do this myself sometimes), I would like to thank you for being my audience. I think that the monarchy of mathematics is geometry and number theory, with geometry being the king of mathematics and number theory being its queen. Geometry is a subject that has interested human beings since ancient times and "differential" geometry is the modern offshoot of classical geometry that studies geometric structures using calculus. In this book I hope that I was able to do justice to a small portion of this field as this is a most beautiful subject with an endless amount of questions that peer deeper and deeper into the imagination of the human mind. And it has many applications in physics, the most famous being Einstein's General Theory of Relativity.

I sincerely hope that I was able to teach you something and I urge you to explore differential geometry further than what you took away from this book. If you have any comments, suggestions, or would like clarification on anything written in this book, please feel free to contact me. I've included my contact information in the preface.

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